{ 2D Projective Geometric Algebra }

2D PGA Cheat Sheet SIGGRAPH 2019 Course Notes

Basics

Basis & Metric:

$$\mathbb{R}^*_{2,0,1}$$

	VECTOR			BIVECTOR			I= PSS
1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{01}$	$\mathbf{e}_{20}$	$\mathbf{e}_{12}$	$\mathbf{e}_{012}$
+1	0	+1	+1	0	0	−1	0
	LINE : ℓ			POINT : P			

Multiplication Table:

1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{01}$	$\mathbf{e}_{20}$	$\mathbf{e}_{12}$	$\mathbf{e}_{012}$
$\mathbf{e}_0$	0	$\mathbf{e}_{01}$	$-\mathbf{e}_{20}$	0	0	$\mathbf{e}_{012}$	0
$\mathbf{e}_1$	$-\mathbf{e}_{01}$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_0$	$\mathbf{e}_{012}$	$\mathbf{e}_2$	$\mathbf{e}_{20}$
$\mathbf{e}_2$	$\mathbf{e}_{20}$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{012}$	$\mathbf{e}_0$	$-\mathbf{e}_1$	$\mathbf{e}_{01}$
$\mathbf{e}_{01}$	0	$\mathbf{e}_0$	$\mathbf{e}_{012}$	0	0	$-\mathbf{e}_{20}$	0
$\mathbf{e}_{20}$	0	$\mathbf{e}_{012}$	$-\mathbf{e}_0$	0	0	$\mathbf{e}_{01}$	0
$\mathbf{e}_{12}$	$\mathbf{e}_{012}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{20}$	$-\mathbf{e}_{01}$	-1	$-\mathbf{e}_0$
$\mathbf{e}_{012}$	0	$\mathbf{e}_{20}$	$\mathbf{e}_{01}$	0	0	$-\mathbf{e}_0$	0

Operators:

$\mathbf{ab}$		Geometric Product	
$\mathbf{a}^*$		Dual	
$\mathbf{a}^\perp$	$\mathbf{a}\mathbf{I}$	Polar	
$\tilde{\mathbf{a}}$		Reverse	
$\hat{\mathbf{a}}$		Normalization	
$\langle\mathbf{a}\rangle_\mathbf{n}$		Select grade n	
$\mathbf{a}\wedge\mathbf{b}$	$\langle\mathbf{ab}\rangle_{\mathbf{s+t}}$	Outer Product	meet
$\mathbf{a}\vee\mathbf{b}$	$(\mathbf{a}^*\wedge\mathbf{b}^*)^*$	Regressive Product	join
$\mathbf{a}\cdot\mathbf{b}$	$\langle\mathbf{ab}\rangle_{\mathbf{\lvert s-t\rvert}}$	Inner Product	
$\mathbf{a}\times\mathbf{b}$	$\frac{1}{2}(\mathbf{ab}-\mathbf{ba})$	Commutator Product	
	$\mathbf{a}\mathbf{b}\tilde{\mathbf{a}}$	Sandwich Product	

Dual, Reverse:

Multivector	$a+b\mathbf{e}_0+c\mathbf{e}_1+d\mathbf{e}_2+e\mathbf{e}_{01}+f\mathbf{e}_{20}+g\mathbf{e}_{12}+h\mathbf{e}_{012}$
Dual	$h+g\mathbf{e}^0+f\mathbf{e}^1+e\mathbf{e}^2+d\mathbf{e}^{01}+c\mathbf{e}^{20}+b\mathbf{e}^{12}+a\mathbf{e}^{012}$
Reverse	$a+b\mathbf{e}_0+c\mathbf{e}_1+d\mathbf{e}_2-e\mathbf{e}_{01}-f\mathbf{e}_{20}-g\mathbf{e}_{12}-h\mathbf{e}_{012}$

Sub-algebras:

$\{1\}$	$\mathbb{R}$	Real	$\{1,\mathbf{e}_{12}\}$	$\mathbb{C}$	Complex
$\{1,\mathbf{e}_0\}$	$\mathbb{D}$	Dual	$\{1,\mathbf{e}_1\}$	$\mathbb{D}$	Hyperbolic
$\{1,\mathbf{e}_{12}\}$		rotors	$\{1,\mathbf{e}_{01},\mathbf{e}_{20}\}$		translators
$\{1,\mathbf{e}_{01},\mathbf{e}_{20},\mathbf{e}_{12}\}$		motors			

Geometry

Points, Lines:

Euclidean point at (x,y) — $x\mathbf{e}_{20}+y\mathbf{e}_{01}+\mathbf{e}_{12}$

Direction (ideal point) (x,y) — $x\mathbf{e}_{20}+y\mathbf{e}_{01}$

Line with eq. $a\mathbf{x}+b\mathbf{y}+c=0$ — $\boldsymbol{\ell}=a\mathbf{e}_1+b\mathbf{e}_2+c\mathbf{e}_0$

Incidence:

Join points $\mathbf{P}_1,\mathbf{P}_2$ in line $\boldsymbol{\ell}$ — $\boldsymbol{\ell}=\mathbf{P}_1\vee\mathbf{P}_2$

Meet lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$ in point P — $P=\boldsymbol{\ell}_1\wedge\boldsymbol{\ell}_2$

Project, Reject:

Line orthogonal to line $\boldsymbol{\ell}$, through point P — $\boldsymbol{\ell}\cdot\mathbf{P}=\boldsymbol{\ell}\times\mathbf{P}$

Project point $\mathbf{P}$ on line $\boldsymbol{\ell}$ — $(\boldsymbol{\ell}\cdot\mathbf{P})\boldsymbol{\ell}$

Project line $\boldsymbol{\ell}$ on point $\mathbf{P}$ — $(\boldsymbol{\ell}\cdot\mathbf{P})\mathbf{P}$

Direction orthogonal to line $\boldsymbol{\ell}$ — $\boldsymbol{\ell}^\perp:=\boldsymbol{\ell}\mathbf{I}$

Norms and numerical values:

Euc. norm of $\boldsymbol{\ell}=c\mathbf{e}_0+a\mathbf{e}_1+b\mathbf{e}_2$: — $\|\boldsymbol{\ell}\|:=\sqrt{\boldsymbol{\ell}^2}(=\sqrt{a^2+b^2})$

Euc. norm of $\mathbf{P}=x\mathbf{e}_{20}+y\mathbf{e}_{01}+z\mathbf{e}_{12}$: — $\|\mathbf{P}\|:=\sqrt{\mathbf{P}\tilde{\mathbf{P}}}\ (=\sqrt{z^2})$

Ideal norm of ideal $\mathbf{P}=x\mathbf{e}_{20}+y\mathbf{e}_{01}$: — $\|\mathbf{P}\|_\infty:=\sqrt{x^2+y^2}$

Norm of motor $\mathbf{m}$ — $\|\mathbf{m}\|:=\sqrt{\mathbf{m}\tilde{\mathbf{m}}}$

Numerical value of ideal $\boldsymbol{\ell}=c\mathbf{e}_0$: — $\|\boldsymbol{\ell}\|_\infty:=c$

Numerical value of pseudoscalar $a\mathbf{I}$ — $\|a\mathbf{I}\|_\infty=a$

Metric:

Distance between points $\mathbf{P}_1,\mathbf{P}_2$ — $\|\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2\|,\ \|\hat{\mathbf{P}}_1\times\hat{\mathbf{P}}_2\|_\infty$

Angle of intersecting lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$ — $\cos^{-1}(\hat{\boldsymbol{\ell}}_1\cdot\hat{\boldsymbol{\ell}}_2),\ \sin^{-1}(\|\hat{\boldsymbol{\ell}}_1\wedge\hat{\boldsymbol{\ell}}_2\|)$

Distance parallel lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$ — $\|\hat{\boldsymbol{\ell}}_1\wedge\hat{\boldsymbol{\ell}}_2\|_\infty$

Oriented dist. eucl. $\mathbf{P}$ to line $\boldsymbol{\ell}$ — $\hat{\mathbf{P}}\vee\hat{\boldsymbol{\ell}},\ \|\hat{\mathbf{P}}\wedge\hat{\boldsymbol{\ell}}\|_\infty$

Angle betw. ideal $\mathbf{P}$ and line $\boldsymbol{\ell}$ — $\sin^{-1}\|\hat{\mathbf{P}}\wedge\hat{\boldsymbol{\ell}}\|_\infty$

Angle bisector of $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ — $(\hat{\boldsymbol{\ell}}_1+\hat{\boldsymbol{\ell}}_2)$ or $\hat{\boldsymbol{\ell}}_1-\hat{\boldsymbol{\ell}}_2$

Perp. bisector of $\mathbf{P}_1$ and $\mathbf{P}_2$ — $(\hat{\mathbf{P}}_1+\hat{\mathbf{P}}_2)(\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2)$

Altitudes of $\Delta\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$ — $(\mathbf{P}_1\vee\mathbf{P}_2)\cdot\mathbf{P}_3$, etc.

Motors

Rotors & Translators:

Rotator α around point $\mathbf{P}_E$ — $e^{\frac{\alpha}{2}\mathbf{P}_E}=\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\mathbf{P}_E$

Translator d orthogonal to $\mathbf{P}_\infty$ — $e^{\frac{d}{2}\mathbf{P}_\infty}=1+\frac{d}{2}\mathbf{P}_\infty$

Motor between lines $\boldsymbol{\ell}_1$,$\boldsymbol{\ell}_2$ — $\sqrt{\hat{\boldsymbol{\ell}}_2\hat{\boldsymbol{\ell}}_1}$

Logarithm of motor $\mathbf{m}$ — $\widehat{\langle\mathbf{m}\rangle_2}$

Compose & Apply:

Compose motors $\mathbf{m}_1$ and $\mathbf{m}_2$ — $\mathbf{m}_2\mathbf{m}_1$

Normalize motor $\mathbf{m}$ — $\hat{\mathbf{m}}=\dfrac{\mathbf{m}}{\|\mathbf{m}\|}$

Square root of motor $\mathbf{m}$ — $\sqrt{\mathbf{m}}=\widehat{(1+\mathbf{m})}$

Reflect element $\mathbf{X}$ in line $\boldsymbol{\ell}$ — $\boldsymbol{\ell}\mathbf{X}\boldsymbol{\ell}$

Transform $\mathbf{X}$ with motor $\mathbf{m}$ — $\mathbf{m}\mathbf{X}\tilde{\mathbf{m}}$

More

Areas:

Area of $\Delta\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$ — $\frac{1}{2}(\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2\vee\hat{\mathbf{P}}_3)$

Length of closed loop $\mathbf{P}_1\mathbf{P}_2\ldots\mathbf{P}_n$ — $\sum_{i=1}^{n-1}\|\hat{\mathbf{P}}_i\vee\hat{\mathbf{P}}_{i+1}\|$

Area of closed loop $\mathbf{P}_1\mathbf{P}_2\ldots\mathbf{P}_n$ — $\frac{1}{2}\|(\sum_{i=1}^{n-1}\hat{\mathbf{P}}_i\vee\hat{\mathbf{P}}_{i+1})\|_\infty$

Rigid Body Mechanics: (Valid in euclidean, elliptic & hyperbolic planes)

Kinematics-points, dynamics-lines — linear+angular unified

Element in the body/space frame — $\mathbf{x}_b/\mathbf{x}_s$

Path of $\mathbf{x}$ under the motion $\mathbf{g}$ — $\mathbf{x}_s=\mathbf{g}\mathbf{x}_b\widetilde{\mathbf{g}},\ \mathbf{x}_b=\widetilde{\mathbf{g}}\mathbf{x}_s\mathbf{g}$

Velocity $\mathbf{V}_b$ in the body — $\mathbf{V}_b=\tilde{\mathbf{g}}\dot{\mathbf{g}}$ (a bivector)

Inertia tensor $\mathsf{A}:\bigwedge^2\leftrightarrow\bigwedge^1$ — maps vel. $\leftrightarrow$ mom. in body

Momentum line $\mathbf{m}_b$ in the body — $\mathbf{m}_b=\mathsf{A}(\mathbf{V}_b)$

Kinetic energy E — $E=\mathbf{m}_b\vee\mathbf{V}_b$

Euler Eq. of Motion 1: — $\dot{\mathbf{g}}=\mathbf{g}\mathbf{V}_b$

Euler EoM 2: ($\mathbf{f}_b$ = ext. forces) — $\dot{\mathbf{V}}_b=2\mathsf{A}^{-1}(\mathbf{f}_b+(\mathbf{m}_b\times\mathbf{V}_b))$

Time derivative of energy E — $\dot{E}=-2\mathbf{f}_b\vee\mathbf{V}_b$

Work $w(t)=E(t)-E(0)$ — $=\int_0^t\dot{E}ds=-2\int_0^t\mathbf{f}_b\vee\mathbf{V}_b ds$