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GAME2020

Geometric Algebra Mini Event

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A new language for Physics

A New Language for Physics

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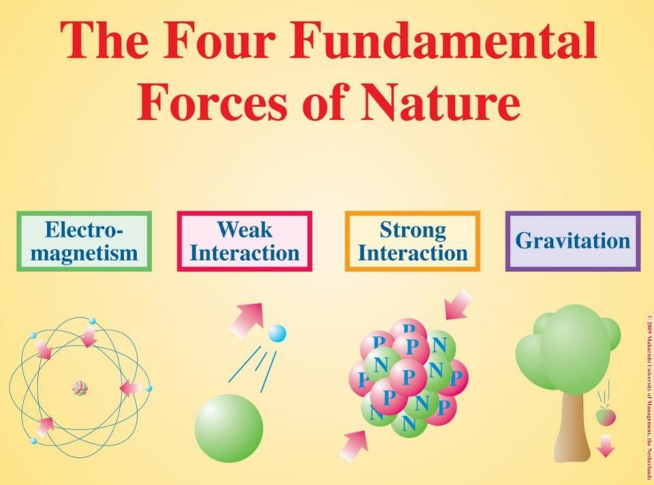
Overview

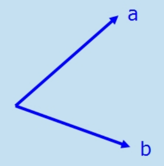
- ‘Geometric Algebra’ is an extremely useful approach to the mathematics of physics, that allows one to use a common language in a huge variety of contexts
- E.g. complex variables, vectors, quaternions, matrix theory, differential forms, tensor calculus, spinors, twistors, are all subsumed under a common approach
- Therefore results in great efficiency — can quickly get into new areas
- Also tends to suggest new geometrical (therefore physically clear, and coordinate-independent) ways of looking at things
- Will try today to introduce a few aspects of it in more detail — principally applications to electromagnetism, quantum mechanics and gravity — in fact will discuss all four forces of nature



- For those who maybe have not been to the preceding lectures, will give a short introduction to what GA is, but aim to move on quickly to the new material with the Spacetime Algebra (STA)
- For further info and pointers to where else it's useful, look at http://geometry.mrao.cam.ac.uk

Geometric Algebra

- Know that for complex numbers there is a 'unit imaginary' i
- Main property is that $i^2 = -1$
- How can this be? (any ordinary number squared is positive)
- Troubled some very good mathematicians for many years
- Usually these days an object with these properties just defined to exist, and 'complex numbers' are defined as $x + iy$ (x and y ordinary numbers)
- But consider following: Suppose have two directions in space



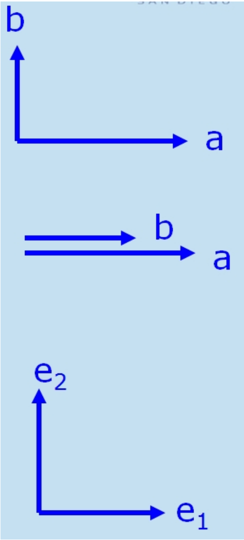
a and b (these are called 'vectors' as usual)

- And suppose we had a language in which we could use vectors as words and string together meaningful phrases and sentences with them So e.g. ab or bab or $abab$ would be meaningful phrases

Geometric Algebra (contd.)

Now introduce two rules:

- If a and b perpendicular, then $ab = -ba$
- If a and b parallel (same sense) then $ab = |a||b|$ (product of lengths)
- Just this does an amazing amount of mathematics!
- E.g. suppose have two unit vectors at right angles
- Rules say $e_1^2 = e_1 e_1 = 1$, $e_2^2 = e_2 e_2 = 1$ and $e_1 e_2 = -e_2 e_1$



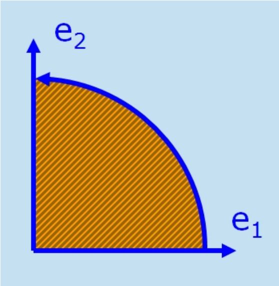
Geometric Algebra (contd.)

Try $(e_1 e_2)^2$

- This is

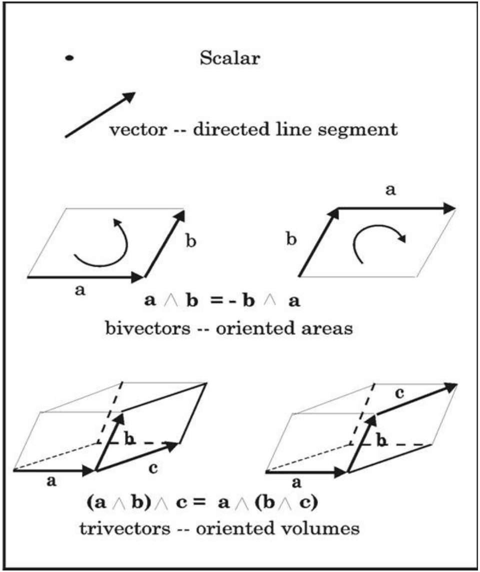
$$e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1$$

- We have found a geometrical object $(e_1 e_2)$ which squares to minus 1 !
- Can now see complex numbers are objects of the form $x + (e_1 e_2) y$
- What is $(e_1 e_2)$? —— we call it a bivector
- Can think of it as an oriented plane segment swept out in going from e_1 to e_2



- More generally, given any two vectors a and b we can form $a \wedge b$ where we sweep out over the angle between them:

An algebra of geometric objects



Geometric Algebra

- Consider a vector space with the usual inner product;

$$\boldsymbol{a}\cdot\boldsymbol{b}$$

The new outer or wedge product produces a new quantity called a bivector

$$\boldsymbol{a}\wedge\boldsymbol{b}$$

Combine these into a single geometric product:

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}$$

Unlike the inner and outer products, this product is **INVERTIBLE**

- Taking instead the geometric product as primary, we have

$$\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba})$$

and

$$\boldsymbol{a}\wedge\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba})$$

- This is basis for axiomatic development

3D Geometric Algebras cont

- In 3D we have three orthonormal basis vectors: $e_1, \; e_2, \; e_3$

$$e_1^2 = 1, \;\; e_2^2 = 1, \;\; e_3^2 = 1, \;\; e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = 0$$

$$e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3 \equiv I$$

- Note I times any vector is a bivector

$$Ie_1 = e_2 e_2, \quad Ie_2 = e_3 e_1, \quad Ie_3 = e_1 e_2$$

- Again, look at the properties of this trivector, I on squaring

$$I^2 = (e_1 e_2 e_3)(e_1 e_2 e_3) = e_1 e_1 e_2 e_3 e_2 e_3 = -(e_1 e_1)(e_2 e_2)(e_3 e_3) = -1$$

- So, we have another real geometric object which squares to -1 ! Indeed there are many such objects which square to -1 ; this means that we seldom have need for complex numbers....
- Call the highest grade object in the space the pseudoscalar – unique up to scale

Reflections

- As seen in earlier talks, reflections are very easy to implement in GA
- Consider reflecting a vector a in a plane with unit normal n , the reflected vector a' is given by:

$$a' = -nan$$

- This can easily be seen via

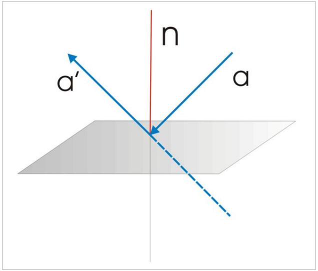
$$2(n \wedge a)n = (na - an)n = nan - a$$

and therefore

$$2(n \wedge a + n \cdot a)n = 2nan$$
$$= 2(a \cdot n)n + nan - a$$

and so

$$-nan = a - 2(n \cdot a)n$$



Rotations

- For many applications rotations are also an extremely important aspect of GA – first consider rotations in 3D:
- Recall that two reflections form a rotation:

$$a \mapsto -m(-nan)m = mnanm$$

- We therefore define our rotor R to be

$$R = mn \quad \text{and rotations are given by} \quad a \mapsto Ra\tilde{R}$$

- Note that this is a geometric product!
- The operation of reversion is the reversing of the order of products, eg

$$\tilde{R} = nm \quad \text{and therefore} \quad R\tilde{R} = 1$$

- Works in spaces of any dimension or signature. Works for all grades of multivectors

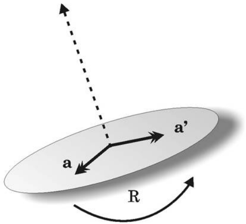
$$A \mapsto RA\tilde{R}$$

Rotations cont...

A rotor, R, is therefore an element of the algebra and can also be written as the exponential of a bivector.

$$R = e^{-B}, \quad B = In\theta/2$$

$$R = \cos\frac{\theta}{2} - In\sin\frac{\theta}{2}$$



The bivector B gives us the plane of rotation (cf Lie groups and quaternions). A rotor is a scalar plus bivector.
Comparing with quaternions

$$q = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \qquad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

$$\boldsymbol{i} = Ie_1, \quad \boldsymbol{j} = -Ie_2 \quad \boldsymbol{k} = Ie_3$$

An Algebra for Spacetime I

- Aim — to construct the geometric algebra of spacetime. Invariant interval is

$$s^2 = c^2 t^2 - \mathrm{x}^2 - \mathrm{y}^2 - \mathrm{z}^2$$

- Work in natural units, $c = 1$.
- Need four vectors $\{e_0, e_i\}, i = 1 \dots 3$ with properties

$$e_0{}^2 = 1, \qquad e_i{}^2 = -1$$
$$e_0 \cdot e_i = 0, \qquad e_i \cdot e_j = -\delta_{ij}$$

- Summarised by

$$e_\mu \cdot e_\nu = \mathrm{diag}(+ - - -) = \text{Minkowski metric}\, \eta_{\mu\nu}, \quad \mu, \nu = 0 \dots 3$$

Bivectors

$4 \times 3/2 = 6$ bivectors in algebra. Two types

1. Those containing e_0, *e.g.* $\{e_i \wedge e_0\}$,
2. Those not containing e_0, *e.g.* $\{e_i \wedge e_j\}$.

An Algebra for Spacetime II

- For any pair of vectors a and b, with $a \cdot b = 0$, have

$$(a \wedge b)^2 = abab = -abba = -a^2 b^2$$

- The two types have different squares
- Spacelike Euclidean bivectors satisfy

$$(e_i \wedge e_j)^2 = -{e_i}^2 {e_j}^2 = -1$$

  and generate rotations in a plane
- Timelike bivectors satisfy

$$(e_i \wedge e_0)^2 = -{e_i}^2 {e_0}^2 = 1$$

  and generate hyperbolic geometry e.g.:

$$\begin{aligned} e^{\alpha e_1 e_0} &= 1 + \alpha e_1 e_0 + \alpha^2/2! + \alpha^3/3!\, e_1 e_0 + \cdots \\ &= \cosh\alpha + \sinh\alpha\, e_1 e_0 \end{aligned}$$

- Crucial to treatment of Lorentz transformations (more below)

The Pseudoscalar

- Define the pseudoscalar I

$$I = e_0 e_1 e_2 e_3$$

- Since I is grade 4, it has

$$\tilde{I} = e_3 e_2 e_1 e_0 = I$$

Compute the square of I :

$$I^2 = I\tilde{I} = (e_0 e_1 e_2 e_3)(e_3 e_2 e_1 e_0) = -1$$

- Multiply bivector by I, get grade $4 - 2 = 2$ — another bivector. Provides map between bivectors with positive and negative square:

$$I e_1 e_0 = e_1 e_0 I = e_1 e_0 e_0 e_1 e_2 e_3 = -e_2 e_3$$

An Algebra for Spacetime IV

- Have four vectors, and four trivectors in algebra. Interchanged by duality

$$e_1 e_2 e_3 = e_0 e_0 e_1 e_2 e_3 = e_0 I = -I e_0$$

- NB I anticommutes with vectors and trivectors. (In space of even dimensions). I always commutes with even-grade.
- Now at this point we settle on a given fixed Cartesian frame of vectors in which to do our physics — can think of this as the laboratory frame, and rename our e_μ to be γ_μ
- Now have available the basic tool for the relativistic physics — the STA

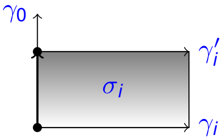
1	$\{\gamma_\mu\}$	$\{\gamma_\mu \wedge \gamma_\nu\}$	$\{I\gamma_\mu\}$	$I = \gamma_0\gamma_1\gamma_2\gamma_3$
1	4	6	4	1
scalar	vectors	bivectors	trivectors	pseudoscalar

An Algebra for Spacetime

- The spacetime algebra or STA. Using the new name $\{\gamma_\mu\}$ for preferred orthonormal frame.
- From $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$ we see that the $\{\gamma_\mu\}$ satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$$

- This is the Dirac matrix algebra!
- Explains notation, but $\{\gamma_\mu\}$ are vectors, not a set of matrices in 'isospace'.
- Each inertial frame defines a set of relative vectors.
- These are spacetime areas swept out while moving along the velocity vector of the frame.



- We define $\sigma_i = \gamma_i \gamma_0$
- These are actually spacetime bivectors, but can function as spatial vectors in the frame orthogonal to γ_0 — call these relative vectors where the relative bit means relative to the velocity vector of the frame
- Easy to show from what we've already defined that they satisfy

$$\begin{aligned} \sigma_i \cdot \sigma_j &= \tfrac{1}{2}(\gamma_i \gamma_0 \gamma_j \gamma_0 + \gamma_j \gamma_0 \gamma_i \gamma_0) \\ &= \tfrac{1}{2}(-\gamma_i \gamma_j - \gamma_j \gamma_i) = \delta_{ij} \end{aligned}$$

and

$$\tfrac{1}{2}(\sigma_i \sigma_j - \sigma_j \sigma_i) = \epsilon_{ijk} I \sigma_k$$

- This is the algebra of the Pauli spin matrices!

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

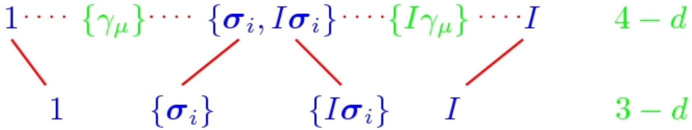
An Algebra for Spacetime VII

- And of course is also the GA of the 3-d relative space in rest frame of γ_0
- A particularly nice feature is that the volume element is

$$\sigma_1\sigma_2\sigma_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = -\gamma_1\gamma_0\gamma_2\gamma_3 = I$$

so the 3-d subalgebra shares same pseudoscalar as spacetime!

- So projected onto the even subalgebra of the STA we have the following picture:



- The 6 spacetime bivectors split into relative vectors and relative bivectors. This split is observer dependent. A very useful technique.

Lorentz Transformations I

- Usually expressed as a coordinate transformation, *e.g.*

$$x' = \gamma(x - \beta t) \quad t' = \gamma(t - \beta x)$$
$$x = \gamma(x' + \beta t') \quad t = \gamma(t' + \beta x')$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and β is scalar velocity.

- Position vector x decomposed in two frames, $\{e_\mu\}$ and $\{e'_\mu\}$,

$$x = x^\mu e_\mu = x^{\mu\prime} e'_\mu$$

- (An aside: Relation of coordinates to these frames comes from the notion of reciprocal frame.
- Given the frame $\{e_\mu\}$ we define the reciprocal frame $\{e^\mu\}$ via

$$e^\mu \cdot e_\nu = \delta^\mu_\nu$$

Lorentz Transformations II

- So e.g. $\gamma^i = -\gamma_i$, ($i = 1, 2, 3$), and $\gamma^0 = \gamma_0$. With these definitions then

$$x^\mu = x \cdot e^\mu$$

  and so $$t = e^0 \cdot x, \qquad t' = e^{0\prime} \cdot x$$

- Very useful for working with curvilinear coordinates in particular, and articulates well with Geometric Calculus)
- Concentrating on the 0, 1 components:

$$t e_0 + \mathrm{x} e_1 = t' e_0' + \mathrm{x}' e_1',$$

- Derive vector relations

$$e_0' = \gamma(e_0 + \beta e_1), \quad e_1' = \gamma(e_1 + \beta e_0).$$

Lorentz Transformations III

- Gives new frame in terms of the old. Now introduce 'hyperbolic angle' α,
$$\mathrm{tanh}\alpha = \beta, \qquad (\beta < 1),$$

- Gives
$$\gamma = (1 - \mathrm{tanh}^2\alpha)^{-1/2} = \mathrm{cosh}\alpha.$$

- Vector e_0' is now
$$\begin{aligned} e_0' &= \mathrm{ch}(\alpha)e_0 + \mathrm{sh}(\alpha)e_1 \\ &= (\mathrm{ch}(\alpha) + \mathrm{sh}(\alpha)e_1 e_0)e_0 = e^{\alpha\, e_1 e_0}\, e_0, \end{aligned}$$

- Similarly, we have
$$e_1' = \mathrm{ch}(\alpha)e_1 + \mathrm{sh}(\alpha)e_0 = e^{\alpha\, e_1 e_0}\, e_1.$$

- Two other frame vectors unchanged. Relationship between the frames is

$$e'_\mu = R e_\mu \tilde{R}, \quad e^{\mu\prime} = R e^\mu \tilde{R}, \quad R = e^{\alpha\, e_1 e_0/2}.$$

- Same rotor prescription works for boosts as well as rotations!
- Spacetime is a unified entity now.
- Generalise this to $R = e^B$ where B is any bivector in the Spacetime Algebra.
- This rotor provides general Lorentz transformations.
- Given any object M in the algebra, we rotate it with $M' = RM\tilde{R}$
- Very simple! Eg.:

The Electromagnetic Field Strength I

- Want to illustrate how useful this version of the Lorentz transformations is in electromagnetism, by looking at the electromagnetic field strength, or Faraday.
- Tensor version is rank-2 antisymmetric tensor $F^{\mu\nu}$. As a matrix, has components

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

- Often see this, but it hides the natural complex structure. In our version, F is a spacetime bivector. We generate it from the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, which are relative vectors in the 3-space orthogonal to the time axis γ_0 (and therefore bivectors in the full STA), via the very simple

$$F = \boldsymbol{E} + I\boldsymbol{B}$$

The Electromagnetic Field Strength II

which we can relate to the matrix elements via

$$F^{\mu\nu} = (\gamma^\nu \wedge \gamma^\mu) \cdot F$$

Since $\gamma_0 F \gamma_0 = (-\boldsymbol{E} + I\boldsymbol{B})$, we can recover $\boldsymbol{E}$ and $I\boldsymbol{B}$ individually from

$$\begin{aligned} \boldsymbol{E} &= \tfrac{1}{2}(F - \gamma_0 F \gamma_0) \\ I\boldsymbol{B} &= \tfrac{1}{2}(F + \gamma_0 F \gamma_0) \end{aligned}$$

- Split into $\boldsymbol{E}$ and $I\boldsymbol{B}$ depends on observer velocity (γ_0). Different observers measure different fields.
- Second observer, velocity $v = R\gamma_0\tilde{R}$, comoving frame $\gamma'_\mu = R\gamma_\mu\tilde{R}$
- Measures components of electric field

$$E'_i = (\gamma'_i \gamma'_0) \cdot F = (R\boldsymbol{\sigma}_i\tilde{R}) \cdot F = \boldsymbol{\sigma}_i \cdot (\tilde{R}FR)$$

- Same transformation law as for vectors. Very efficient. E.g.

The Electromagnetic Field Strength III

- Stationary charges in γ_0 frame set up field

$$F = \boldsymbol{E} = E_x\sigma_1 + E_y\sigma_2$$

- Second observer, velocity $\tanh\alpha$ in γ_1 direction, so

$$R = e^{\alpha\sigma_1/2}$$

- Measures the $\boldsymbol{\sigma}_i$ components of

$$\tilde{R}FR = e^{-\alpha\sigma_1/2}Fe^{\alpha\sigma_1/2} = E_x\sigma_1 + E_y e^{-\alpha\sigma_1}\sigma_2$$

- Gives

$$E'_x = E_x, \quad E'_y = \mathrm{ch}(\alpha)E_y, \quad B'_z = -\mathrm{sh}(\alpha)E_y$$

- If you've ever tried this in the conventional approach, you'll know the approach here is much simpler than working with tensors!

The derivative

- To get further in physics and electromagnetism, we need an additional entity, a derivative operator
- We define this via the reciprocal frame discussed above
- If $\{e_\mu\}$ is a frame, then the reciprocal frame $\{e^\nu\}$ is defined by

$$e^\nu \cdot e_\mu = \delta^\nu_\mu$$

- Note these are vectors, just like the e_μ — don't belong to e.g. a 1-form space
- With $e_\mu = \gamma_\mu$ use these to define the vector differential operator for spacetime

$$\nabla \equiv \gamma^\mu \frac{\partial}{\partial x^\mu} \equiv \gamma^\mu \partial_\mu$$

- (As a quick encouragement, with just this object and the STA, can basically do all of Electromagnetism (EM) and Quantum mechanics through to Electroweak theory without introducing any further new mathematics!)

The derivative contd.

- The definition of ∇ above did not need to use rectangular coordinates and the orthonormal system $\{\gamma_\mu\}$
- Instead, suppose we have an arbitrary coordinate system $\{x^\mu\}$, e.g. standard polars in spacetime (t, r, θ, ϕ)
- If we let x be the 4-d position vector, then the following two frames are reciprocal

$$e_\mu = \partial_\mu x, \quad e^\nu = \nabla x^\nu, \quad \text{i.e. they satisfy} \quad e^\nu \cdot e_\mu = \delta^\nu_\mu$$

  Note the geometric object ∇ is given by $e^\mu \frac{\partial}{\partial x^\mu} = e^\mu \partial_\mu$ in this system
- Secondly, for any vector field $a(x)$, the upstairs and downstairs components are just

$$a^\mu \equiv a \cdot e^\mu \quad \text{and} \quad a_\mu = a \cdot e_\mu$$

- These statements look trivial, but are enough to do everything associated with vector calculus in curvilinear coordinate systems
- Again, this is a great saving in time and effort

Electromagnetism I

- Relative to a particular frame with timelike velocity γ_0, the spactime vector derivative

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^\mu \partial_\mu$$

splits as

$$\nabla \gamma_0 = \partial_t - \boldsymbol{\sigma}_i \partial_i = \partial_t - \boldsymbol{\nabla}$$

where $\boldsymbol{\nabla}$ is the relative (3-d) vector derivative

- Maxwell's equations are:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} = \partial_t (I\boldsymbol{B}) \qquad \boldsymbol{\nabla} \wedge \boldsymbol{B} = I(\boldsymbol{J} + \partial_t \boldsymbol{E})$$

- Using the geometric product, these reduce to

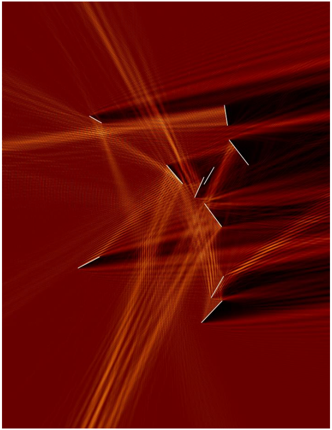
$$\boldsymbol{\nabla}(\boldsymbol{E} + I\boldsymbol{B}) + \partial_t (\boldsymbol{E} + I\boldsymbol{B}) = \rho - \boldsymbol{J}$$

Electromagnetism

- If we define the Lorentz-covariant field strength $F = \boldsymbol{E} + I\boldsymbol{B}$ and current $J = (\rho + \boldsymbol{J})\gamma_0$, we obtain the single, covariant equation

$$\boxed{\nabla F = J}$$

- The advantage here is not merely notational - just as the geometric product is invertible, unlike the separate dot and wedge product, the geometric product with the vector derivative is invertible (via Green's functions) where the separate divergence and curl operators are not
- This led to the development of a new method for calculating EM response of conductors to incoming plane waves
- Was possible to change the illumination in real time and see the effects

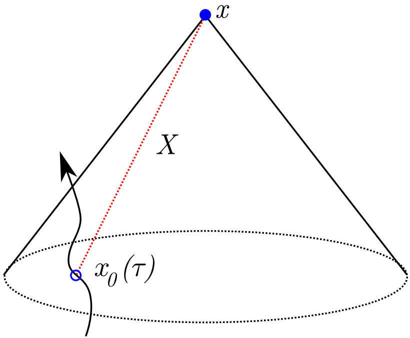


Electromagnetism

- For more detailed example want to consider radiation from a moving charge
- David Hestenes pioneered the techniques on this some years ago
- Think final form for the Faraday is the most compact, and informative, of any that have been achieved
- Since $\nabla \wedge F = 0$, we can introduce a vector potential A such that $F = \nabla \wedge A$
- If we impose $\nabla \cdot A = 0$, so that $F = \nabla A$, then A obeys the wave equation

$$\nabla F = \nabla^2 A = J$$

Point Charge Fields



- Since radiation doesn't travel backwards in time, we have the electromagnetic influence propagating along the future light-cone of the charge.

- An observer at x receives an influence from the intersection of their past light-cone with the charge's worldline, x_0, so the separation vector down the light-cone $X = x - x_0$ is null.
- Fully covariant expression for the Liénard-Wiechert potential is then found to be

$$A = \frac{q}{4\pi} \frac{v}{X \cdot v}$$

- This is true no matter how complicated the motion

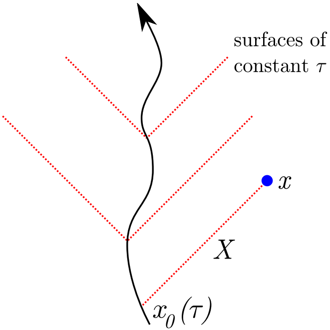
Point Charge Fields

- Now we want to find $F = \nabla A$
- One needs a few differential identities
- Following is perhaps most interesting
- Since $X^2 = 0$,

$$\begin{aligned} 0 = \mathring{\nabla}(\mathring{X} \cdot X) &= \mathring{\nabla}(\mathring{x} \cdot X) - \mathring{\nabla}(\mathring{x_0}(\tau) \cdot X) \\ &= X - \gamma^\mu (X \cdot \partial_\mu x_0(\tau)) \\ &= X - \gamma^\mu (X \cdot (\partial_\mu \tau) \partial_\tau x_0) \\ &= X - (\nabla \tau)(X \cdot v) \end{aligned}$$

$$\Rightarrow \quad \nabla \tau = \frac{X}{X \cdot v} \quad (*)$$

where we treat τ as a scalar field, with its value at $x_0(\tau)$ being extended over the charge's forward light-cone



- Proceeding using this result, and defining

$$\Omega_v = \dot{v} \wedge v$$

which is the acceleration bivector, then result for F itself can be found relatively quickly

Point Charge Fields

$$F = \frac{q}{4\pi} \frac{X \wedge v + \frac{1}{2} X \Omega_v X}{(X \cdot v)^3}$$

- Equation displays clean split into Coulomb field in rest frame of charge, and radiation term

$$F_{rad} = \frac{q}{4\pi} \frac{\frac{1}{2} X \Omega_v X}{(X \cdot v)^3}$$

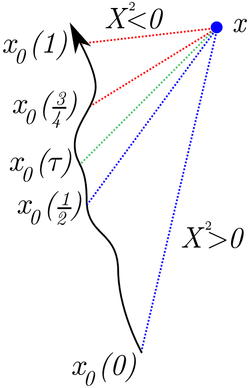
proportional to rest-frame acceleration projected down the null vector X.

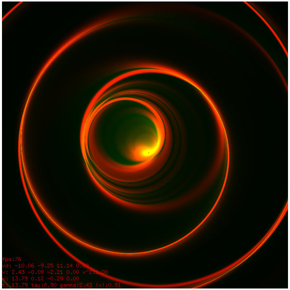
- $X \cdot v$ is distance in rest-frame of charge, so F_{rad} goes as $1/\mathrm{distance}$, and energy-momentum tensor $T(a) = -\frac{1}{2} F a F$ drops off as $1/\mathrm{distance}^2$. Thus the surface integral of T doesn't vanish at infinity - energy-momentum is carried away from the charge by radiation.

Point Charge Fields

For a numerical solution:

- Store particle's history (position, velocity, acceleration)
- To calculate the fields at x, find the null vector X by bisection search (or similar)
- Retrieve the particle velocity, acceleration at the corresponding τ - above formulae give us A and F





- The algebraic structure of wave mechanics arises naturally from the geometric algebra of spacetime
- Allows us to reformulate standard QM in more geometrical way
- Also suggests new lines of interpretation ...
- Look at Pauli spinors.

This works conventionally by regarding the Pauli matrices as being matrix operators on column vectors, the latter being the Pauli spinors. Pauli matrices are

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrix operators (with hats). The $\{\hat{\sigma}_k\}$ act on 2-component Pauli spinors

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

ψ_1, ψ_2 complex
$|\psi\rangle$ in two-dimensional complex vector space

- In GA approach, something rather remarkable happens, we can replace both objects (operators and spinors), by elements of the same algebra. Thus spacetime objects, and relations between them, can replace all (single particle) quantum statements!
- Crucial aspect we have to understand is how to model the Pauli and Dirac spinors within STA. For Pauli spinors (2 complex entries in the column spinor), we put $\psi_1 = a^0 + ia^3$, $\psi_2 = -a^2 + ia^1$ ($a^0, \ldots, a^3$ real scalars) and then the translation (conventional on left, STA on right) is

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I\sigma_k \tag{1}$$

- For spin-up $|+\rangle$, and spin-down $|-\rangle$ get

$$|+\rangle \leftrightarrow 1 \qquad |-\rangle \leftrightarrow -I\sigma_2 \tag{2}$$

Quantum Theory III

- Action of the quantum operators $\{\hat{\sigma}_k\}$ on states $|\psi\rangle$ has an analogous operation on the multivector ψ:

$$\hat{\sigma}_k|\psi\rangle \ \leftrightarrow \ \sigma_k \psi \sigma_3 \quad (k = 1, 2, 3).$$

σ_3 on the right-hand side ensures that $\sigma_k \psi \sigma_3$ stays in the even subalgebra

- Verify that the translation procedure is consistent by computation; e.g.

$$\hat{\sigma}_1|\psi\rangle = \begin{pmatrix} -a^2 + ia^1 \\ a^0 + ia^3 \end{pmatrix}$$

translates to

$$-a^2 + a^1 I\sigma_3 - a^0 I\sigma_2 + a^3 I\sigma_1 = \sigma_1 \psi \sigma_3.$$

- Also need translation for multiplication by the unit imaginary i. Do this via noting

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

- See multiplication of both components of $|\psi\rangle$ achieved by multiplying by the product of the three matrix operators.
- Therefore arrive at the translation

$$i|\psi\rangle \;\leftrightarrow\; \sigma_1\sigma_2\sigma_3\psi(\sigma_3)^3 = \psi I\sigma_3.$$

- Unit imaginary of quantum theory is replaced by right multiplication by the bivector $I\sigma_3$. (Same happens in Dirac case.)
- Now define the scalar

$$\rho \equiv \psi\tilde{\psi}.$$

- The spinor ψ then decomposes into

$$\psi = \rho^{1/2} R,$$

where $R = \rho^{-1/2}\psi$.

- The multivector R satisfies $R\tilde{R} = 1$, so is a rotor. In this approach, Pauli spinors are simply unnormalised rotors!
- Turns out Hermitian adjoint corresponds to reversion followed by reflection in time axis in general, so in 3d have the simple relation: $\psi^\dagger = \tilde{\psi}$
- Quantum inner product is given by

$$\langle\psi|\phi\rangle \leftrightarrow \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3$$

which projects out the 1 and $I\boldsymbol{\sigma}_3$ components of $\psi^\dagger\phi$

- (Note the angle brackets in the GA algebra on the right are instructions to 'take the scalar part'.)

This view offers a number of insights.

- Expectation value of spin in k-direction is

$$\begin{aligned}\langle\psi|\hat{\sigma}_k|\psi\rangle &\leftrightarrow \langle\psi^\dagger\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3\rangle - \langle\psi^\dagger\boldsymbol{\sigma}_k\psi I\rangle I\boldsymbol{\sigma}_3 \\ &= \langle\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3\psi^\dagger\rangle\end{aligned}$$

since $\psi^\dagger\boldsymbol{\sigma}_k\psi$ is a vector.

- Defining the spin vector,

$$\boldsymbol{s} = \frac{1}{2}\hbar\psi\boldsymbol{\sigma}_3\psi^\dagger$$

this reduces to

$$\langle\psi|\hat{s}_k|\psi\rangle \leftrightarrow \frac{1}{2}\hbar\langle\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3\psi^\dagger\rangle = \boldsymbol{\sigma}_k\cdot\boldsymbol{s}$$

So “forming the expectation value of the s_k operator” reduces to projecting out the $\boldsymbol{\sigma}_k$ component of the vector $\boldsymbol{s}$

- Using the 'scaled rotor' deomposition of ψ above we have

$$\boldsymbol{s} = \frac{1}{2}\hbar\rho R\sigma_3\tilde{R}.$$

The double-sided construction of the expectation value contains an instruction to rotate the fixed σ_3 axis into the spin direction and dilate it

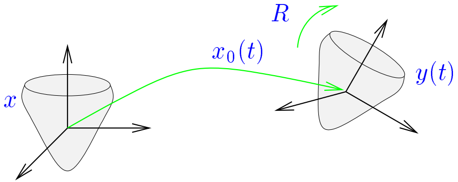
- Also, suppose that the vector $\boldsymbol{s}$ is to be rotated to a new vector $R_0\boldsymbol{s}\tilde{R}_0$. The rotor group combination law tells us that R transforms to R_0R.
  This induces the spinor transformation law

$$\psi \mapsto R_0\psi.$$

This explains the 'spin-1/2' nature of spinor wave functions

- Picking out $\boldsymbol{\sigma}_3$ doesn't break the rotational symmetry of the theory. In rigid-body dynamics, we often choose an arbitrary reference configuration, and formulate the dynamics in terms of the transformation needed to rotate this configuration to the physical one

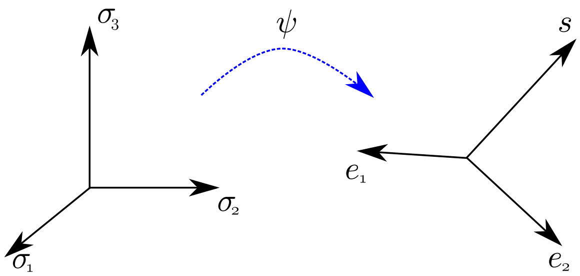
- Relate the vector position of points in the moving body $y(t)$ back to a fixed 'reference' body.



- We let x_0 be the position in space of the centre of mass. Have

$$y(t) = R(t)x\tilde{R}(t) + x_0(t)$$

- Places the rotational motion in the time-dependent rotor $R(t)$.
- The situation quantum mechanically is analogous - we could have chosen any constant vector, and made ψ so that it transformed this into ***s***

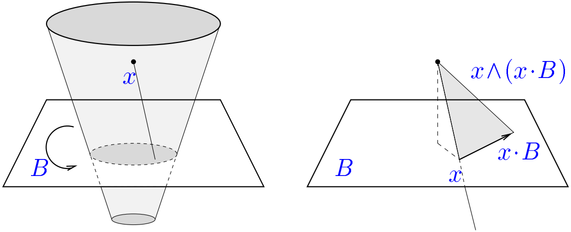


- Worth spending a bit more time on this analogy. Let's look at what happens next in rigid body theory in GA, and then relate typical solutions to QM

We extract inertia tensor $\mathcal{I}$

$$\mathcal{I}(B) = \int d^3x \, \rho x \wedge (x \cdot B)$$

A linear function mapping bivectors to bivectors.



The body rotates in the B plane, at angular frequency $|B|$. The momentum density is $\rho x\cdot B$. Angular momentum density is $x\wedge(\rho x\cdot B)$. Integrate to get the total, $\mathcal{I}(B)$, expressed in the reference body. Rotate to

$$L = R\mathcal{I}(\Omega_B)\tilde{R}$$

Very quick to show that The torque-free equation $\dot{L} = 0$ reduces to

$$\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B) = 0 \quad (*)$$

where we have introduced the extremely useful commutator product

$$A \times B = \tfrac{1}{2}(AB - BA)$$

- So (*) is the Euler equations for rigid body motion
- The usefulness of the GA approach extends not just to the quick derivation, but to solutions.
- Align the body frame $\{e_k\}$ with the principal axes, with moments of inertia $i_k, k = 1 \ldots 3$.
- If we define two constant precession rates and directions

$$\Omega_l = \frac{1}{i_1} L, \qquad \Omega_r = \omega_3 \frac{i_1 - i_3}{i_1} I e_3$$

the Euler equation becomes the rotor equation

$$\dot{R} = -\tfrac{1}{2}\Omega_l R - \tfrac{1}{2} R \Omega_r$$

which integrates immediately to

$$R(t) = \exp(-\tfrac{1}{2}\Omega_l t) R(0) \exp(-\tfrac{1}{2}\Omega_r t)$$

- Fully describes the motion of a symmetric top. An 'internal' rotation in the $e_1 e_2$ plane (a symmetry of the body), followed by a rotation in the angular-momentum plane
- Example video illustrating this is by Christian Perwass
- Now come back to quantum mechanics, and suppose that the particle is placed in a magnetic field, and that all of the spatial dynamics has been separated out.
- Conventionally we introduce the Hamiltonian operator

$$\hat{H} = -\frac{1}{2}\gamma\hbar B_k \hat{\sigma}_k = -\hat{\mu}_k B_k.$$

- The spin state at time t is then written as

$$|\psi(t)\rangle = \alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle,$$

with α and β general complex coefficients

- The dynamical equation for these coefficients is given by the time-dependent Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{d|\psi\rangle}{dt}.$$

- This equation can be hard to analyse, conventionally, because it involves a pair of coupled differential equations for α and β
- Instead, let us see what the Schrödinger equation looks like in the geometric algebra formulation.
- Get the very simple

$$\dot{\psi} = \frac{1}{2}\gamma B_k I\sigma_k\psi = \frac{1}{2}\gamma I\boldsymbol{B}\psi,$$

where $\boldsymbol{B} = B_k\boldsymbol{\sigma}_k$

- If we now decompose ψ into $\rho^{1/2}R$ we see that

$$\dot{\psi}\psi^\dagger = \frac{1}{2}\dot{\rho} + \rho\dot{R}R^\dagger = \frac{1}{2}\rho\gamma I\boldsymbol{B}.$$

- The right-hand side is a bivector, so ρ must be constant. This is to be expected, as the evolution should be unitary. The dynamics now reduces to

$$\dot{R} = \frac{1}{2}\gamma I \boldsymbol{B} R,$$

so the quantum theory of a spin-$1/2$ particle in a magnetic field reduces to a simple rotor equation

- Recovering a rotor equation explains the difficulty of the traditional analysis based on a pair of coupled equations for the components of $|\psi\rangle$
- This approach fails to capture the fact that there is a rotor underlying the dynamics
- Solutions are simple as well
- Suppose that $\boldsymbol{B}$ is a constant field. The rotor equation integrates immediately to give

$$\psi(t) = e^{\gamma I \boldsymbol{B} t/2}\psi_0.$$

- The spin vector $\boldsymbol{s}$ therefore just precesses in the $I\boldsymbol{B}$ plane at a rate $\omega_0 = \gamma|\boldsymbol{B}|$
- Even this simple result is rather more difficult to establish when working with the components of $|\psi\rangle$
- Now let's make the transition to relativistic theory

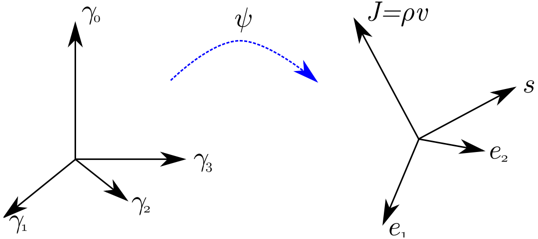
Dirac Theory

- In the relativistic theory of spin-$\frac{1}{2}$ particles, things are similar
- We'll look later at the specific nature of the Dirac wavefunction, but basically it now consists of all elements of the even subalgebra of the STA
- Instead of the wavefunction being a weighted spatial rotor, it's now a full Lorentz spinor:

$$\psi = \rho^{1/2} e^{I\beta/2} R$$

with the addition of a slightly mysterious β term related to antiparticle states

- Five observables in all, including the current, $J = \psi\gamma_0\psi = \rho R\gamma_0\tilde{R}$, and the spin vector $s = \psi\gamma_3\psi = \rho R\gamma_3\tilde{R}$



- The wavefunction obeys the Dirac equation. Here it is in the form first proposed by David Hestenes:

$$\nabla\psi I\sigma_3 - eA\psi = m\psi\gamma_0$$

- This implies that the current J is conserved,

$$\nabla \cdot J = 0$$

with the implication that a fermion cannot be created or destroyed (pair annihilation / production are high-energy multiparticle processes, not covered by the Dirac equation)

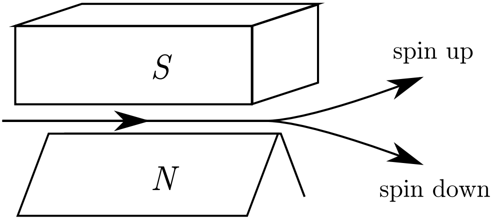
- The timelike component of J is positive definite, and is interpreted as a probability density: a normalised wavefunction has

$$\int d^3x \; J_0 = 1$$

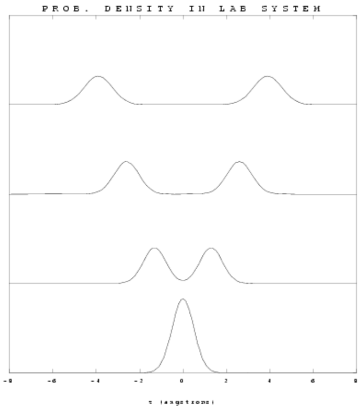
- Conservation of J implies that the probability density “flows” along non-intersecting streamlines - useful for visualisation.

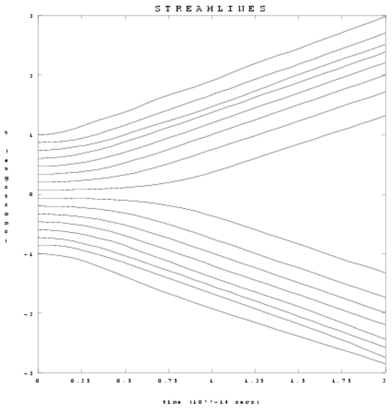
Dirac Theory

- A sample application - Stern-Gerlach apparatus
- Apply a delta-function magnetic field gradient to simulate the apparatus, and numerically calculate the effect of this shock on a wave-packet, with spin initially orthogonal to the magnetic field
- Result : the wave-packet splits into two parts, spins aligned/anti-aligned with the magnetic shock, with streamlines bifurcating depending on where they are in the wave-packet
- Instead of viewing the device as 'measuring' the spin in z direction, and obtaining one of two eigenstates, the apparatus acts as a spin polariser, forcing the spins to align with the magnetic shock

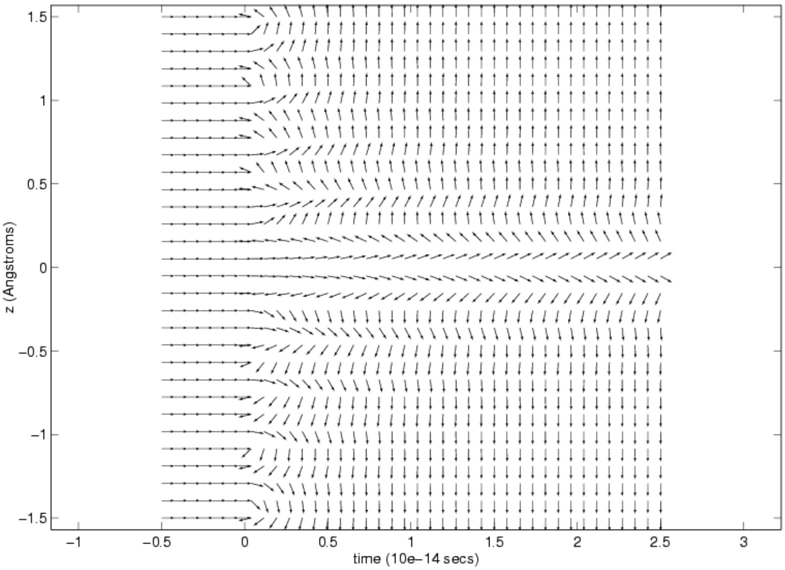


Dirac Theory





Dirac Theory



Gauge Theory Gravity

- So next we get to gravity!
- Want to consider a version of gravity that aims to be as much like our best descriptions of the other 3 forces of nature:
  - The strong force (nuclei forces)
  - The weak force (e.g. radioactivity etc.)
  - electromagnetism
- These are all described in terms of Yang-Mills type gauge theories (unified in quantum chromodynamics) in a flat spacetime background
- In the same way, Gauge Theory Gravity (GTG) is expressed in a flat spacetime
- The key question is what we are gauging. We choose this to be Lorentz rotations at a point, and the ability to carry out an arbitrary remapping from one spacetime point to another
- Why should we want this?
- Find that the Dirac equation and Dirac spinors is probably the easiest place to start

Gauge Theory Gravity

- Take spinors (i.e. Dirac wavefunctions) $\psi_1(x)$ and $\psi_2(x)$. A sample physical statement is

$$\psi_1(x) = \psi_2(x)$$

  i.e. at a point where one field has a particular value, the second field has the same value.

- This is independent of where we place the fields in the STA. Could equally well introduce two new fields

$$\psi_1'(x) = \psi_1(x'), \quad \psi_2'(x) = \psi_2(x'),$$

  with x' an arbitrary function of x. Equation $\psi_1'(x) = \psi_2'(x)$ has precisely the same physical content as original

- Same is true if act on fields with a spacetime rotor

$$\psi_1' = R\psi_1, \quad \psi_2' = R\psi_2$$

  Again, $\psi_1' = \psi_2'$ has same physical content as original equation

- Only thing for which this doesn't work is derivatives

Gauge Theory Gravity

- E.g., suppose R in rotation case is a function of position
- then

$$\nabla(R\psi) = (\nabla R)\psi + \dot{\nabla} R \dot{\psi} \neq R(\nabla\psi)$$

  (here the dots indicate what the ∇ is operating on). We've failed to achieve a covariant operation in at least two ways
- Also position remapping won't work with derivatives, since if $x \mapsto f(x)$ (we call this a position gauge change), then it turns out that

$$\nabla_x \phi'(x) = \bar{f}(\nabla_{x'} \phi(x'))$$

  where the linear function $f(a)$ is given by $f(a) = a\cdot\nabla f(x)$, and $\bar{f}$ denotes the adjoint function.
- I.e. an extraneous $\bar{f}$ gets in the way of covariance here
- We solve all these problems by introducing two gauge fields $\bar{h}(a)$ — a **vector** field — and $\Omega(a)$ — a **bivector** field

Gauge Theory Gravity (contd.) I

- $\overline{h}(a)$: this is defined to have the transformation property $\overline{h}(a) \mapsto \overline{h}\left(\overline{f}^{-1}(a)\right)$ under the position gauge change, so it's able to soak up the extraneous $\overline{f}$ if we use it to 'protect' each derivative operator ∇, i.e. we henceforth use $\overline{h}(\nabla)$ instead of ∇
- $\Omega(a)$: this allows Lorentz rotations (e.g. like $\psi \mapsto R\psi$) to be gauged locally (rotation gauge change). The transformation property needed for this is

$$\Omega(a) \mapsto \Omega'(a) = R\Omega(a)\tilde{R} - 2a\cdot\nabla R\tilde{R}$$

- Covariant derivative in a direction (for a quantity transforming double-sidedly) is

$$\mathcal{D}_a \equiv a\cdot\nabla + \Omega(a)\times$$

where the $\times$ means the GA commutator product

$$A\times B \equiv \frac{1}{2}(AB - BA)$$

- Turns out that the properties of the $\times$ operator (basically, it satisfies Jacobi identity) together with fact $\Omega(a)$ is a bivector mean that $\mathcal{D}_a$ is a scalar operator and satisfies Leibniz rule for derivatives
- Get full vector covariant derivative via $\mathcal{D} \equiv \bar{h}(\partial_a)\mathcal{D}_a$
- ∂_a is the multivector derivative w.r.t. a
- Haven't got time for details, but multivector derivative by a vector a, given, in a frame in which $a = a^\mu e_\mu$, by

$$\partial_a \equiv e^\mu \frac{\partial}{\partial a^\mu}$$

- For a general n-d space, and acting on a grade-r object, these satisfy

$$\partial_a a \cdot A_r = r A_r$$
$$\partial_a a \wedge A_r = (n - r) A_r$$

and $$\boxed{\partial_a A_r a = (-1)^r (n - 2r) A_r}$$

- Note the last of these means that if we differentiate a vector through a bivector, in 4d, the result vanishes. Not obvious, but this turns out to be the key to why e.g. electromagnetism is a massless theory in 4d, and also being able to demonstrate how the Riemann tensor for a black hole works! (see shortly)

Gauge Theory Gravity (contd.) IV

- Returning to gravity, the field strength tensor got by commuting covariant derivatives:

$$[\mathcal{D}_a, \mathcal{D}_b]M = R(a\wedge b)\times M \qquad (M \text{ some multivector field})$$

- This leads to the Riemann tensor

$$R(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b)$$

  from which we make a fully covariant version $\mathcal{R}(B) = Rh(B)$
- Note this is a mapping of bivectors to bivectors — just like the inertia tensor!
- Ricci scalar (rotation gauge and position gauge invariant) is

$$\mathcal{R} = (\partial_b\wedge\partial_a)\cdot\mathcal{R}(a\wedge b)$$

- Simplest gravitational action to use is then $\mathcal{L}_{\rm grav} = \det h^{-1}\mathcal{R}$

Gauge Theory Gravity (contd.) V

- The dynamical variables are $\overline{h}(a)$ and $\Omega(a)$ and field equations correspond to taking $\partial_{\overline{h}(a)}$ and $\partial_{\Omega(a)}$
- That's it! Further details and full description in Lasenby, Doran & Gull, Phil.Trans.Roy.Soc.A. (1998), **356**, 487
- Now what theory do we obtain in this way?
- Locally, theory reproduces predictions of an extension of General Relativity (GR) known as Einstein-Cartan theory (incorporates quantum spin)
- Differs on global issues such as nature of horizons, and topology
- Advantages of GTG include being clear about what the physical predictions of the theory are (since a gauge theory) — they are the quantities that are gauge-invariant!
- Conceptually simpler than standard GR (since works in a flat space background)

- Also simpler in a practical sense, since if you have a computer algebra program that can do Clifford algebra in spacetime, then you can immediately start exploring gravity — this is a lot of fun! — don't need a tensor calculus program, or indeed any extension to curved space!
- Also articulates very well with Dirac equation (in Hestenes form)

What do the equations of motion and solutions look like?

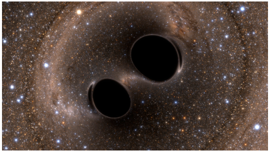
- Equations of motion in absence of matter are

$$\boxed{\partial_a \mathcal{R}(a \wedge b) = 0, \quad \mathcal{D} \wedge \bar{h}(a) = 0}$$

Pretty simple!

- All the symmetries of the Riemann that one encounters conventionally are encoded in the $\partial_a \wedge \mathcal{R}(a \wedge b) = 0$ part of the first equation, and the second equation effectively says that the torsion vanishes in this case

Black holes



- So how do black holes look in this approach?
- Just like setting up EM equations for a point charge, we need to choose a gauge and work from there
  - We would like a gauge (choice of $\overline{h}$-function) that covers all of (flat) space, except possibly a singularity at the origin
  - Again, just like EM, we would expect the field strength tensor to be the same, and not depend on our choice of gauge
  - Denoting e_r as the unit radial vector, $e_t = \gamma_0$ as the unit time vector and the radial null vector $e_- = e_t - e_r$, then two good choices for $\overline{h}$ are

$$\overline{h}(a) = a - \sqrt{\frac{2M}{r}}(a{\cdot}e_r)e_t \qquad \text{and} \qquad \overline{h}(a) = a + \frac{M}{r}(a{\cdot}e_-)e_-$$

We call the first the Newtonian gauge since a lot of the physics looks very Newtonian-like in this gauge, and the second is the GTG analogue of the Advanced Eddington-Finkelstein metric (good for treating photons)

Black holes

- Both are pretty simple!
- They both lead to the Riemann tensor

$$\mathcal{R}(B) = -\frac{M}{2r^3}(B + 3\sigma_r B \sigma_r)$$

  where $\sigma_r = e_r e_t$ is the unit spatial bivector in the radial direction
- We can immediately check the field equation $\partial_a \mathcal{R}(a \wedge b) = 0$ is satisfied
- Using the results for the ∂_a derivative above, we have

$$\partial_a (a \wedge b + 3\sigma_r (a \wedge b)\sigma_r) = 3b + 3\partial_a (\sigma_r (ab - a \cdot b)\sigma_r) = 3b - 3b\sigma_r^2 = 0$$

  where the result that differentiating a vector through a bivector gives zero (here $\dot{\partial}_a \sigma_r \dot{a} = 0$), is a crucial step
- This is quite impressive as regards compactness and ease of working
- Even more impressive is doing the same for a rotating black hole — the Kerr solution

Rotating black holes

- Here if the black hole has angular momentum parameter L, we find

$$\mathcal{R}(B) = -\frac{M}{2\,(r + IL\cos\theta)^3}\,(B + 3\sigma_r B \sigma_r)$$

  i.e. we get from the Schwarzschild (non-rotating) black hole via $r \mapsto r + IL\cos\theta$
- Explains the complex structure previously noticed in the Kerr solution, but in terms of the spacetime pseudoscalar I
- Notice we don't need to do any more work to show that $\partial_a \mathcal{R}\,(a \wedge b) = 0$ is satisfied — follows from what we did in the Schwarzschild case, since $\partial_a I = -I\partial_a$
- Of course quite a lot of work necessary to get from an $\overline{h}$-function to Riemann in this case, but this is certainly the most compact form of Riemann for the Kerr I've ever seen (most authors don't even try to write down the Riemann components!)
- Also, using GA methods, Chris Doran was able to find a compact $\overline{h}$-function gauge for the Kerr which is similar to the Newtonian gauge form for Schwarzschild — metric form of this known as the Doran metric

The GTG approach to gravitational waves I

- Going to give a very abbreviated version of this!
- Start with a plane analogue of the Advanced Eddington Finklestein $\overline{h}$-function discussed above: $\overline{h}(a) = a + \frac{M}{r}(a \cdot e_-)e_-$, where $e_- = e_t - e_r$
- For a gravitational wave propagating in the z direction, the answer is

$$\boxed{\overline{h}(a) = a - \tfrac{1}{2} H \, a \cdot e_+ \, e_+}$$

  where $e_+ = e_t + e_z$, and $H = H(t, x, y, z)$ is a scalar function of spacetime position
- This is all we need!
- A remarkable feature is that despite being very simple, this provides an exact solution for gravitational waves
- With the ansatz $H(t, x, y, z) = G(\eta) f(x, y)$, where $\eta \equiv t - z$, find that $\partial_a \mathcal{R}(a \wedge b) = 0$ is satisfied provided the 2d Laplacian $\nabla^2 f = 0$

The GTG approach to gravitational waves II

- Using polar coords (ρ, ϕ) for the 2d (x, y) plane, the solutions of $\nabla^2 f = 0$ that are picked out as giving homogenous values for the Riemann (i.e. the same all over the plane wavefront) are

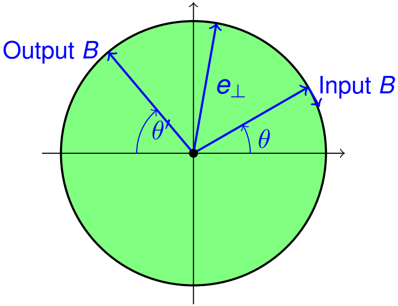
$$f = \rho^2 \cos 2\phi \quad \text{and} \quad f = \rho^2 \sin 2\phi$$

- Borrowing some freedom from $G(\eta)$, we get the final form of Riemann:

$$\boxed{\mathcal{R}(B) = \tfrac{1}{2} G(\eta)\,(e_+ e_\perp) B (e_+ e_\perp)}$$

  where $e_\perp = \cos(\phi_0(\eta)) e_x + \sin(\phi_0(\eta)) e_y$ is the arbitrary polarization direction in the (x, y) plane
- This is very neat in showing us how the input bivector B is reflected in the bivector $e_+ e_\perp$, which encodes both the direction of propagation in spacetime, and the direction of polarization

- The way that the polarization angle is given by ϕ_0, whereas the solution for H and the components of the Riemann rotate through $2\phi_0$, is of course a consequence of the 'spin-2' nature of gravitational radiation, and it is interesting to see it arising here due to the fact we are *reflecting* in the polarization direction



$\theta' = \theta + 2\phi_0$

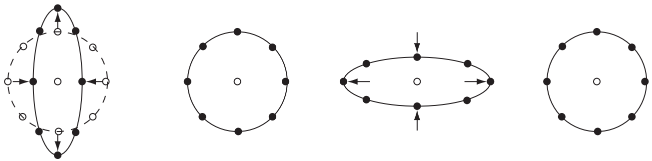
- Notice how simple it is to see that the field equation $\partial_a \mathcal{R}(a\wedge b) = 0$ is satisfied
- The 'pulse' $G(\eta)$ is just a scalar term, so we need

$$\partial_a (e_+ e_\perp (a\wedge b) e_\perp e_+) = \partial_a (e_+ e_\perp (ab - a\cdot b) e_\perp e_+)$$
$$= -b e_+ e_\perp e_\perp e_+ = b e_+ e_+ = 0$$

since e_+ is null

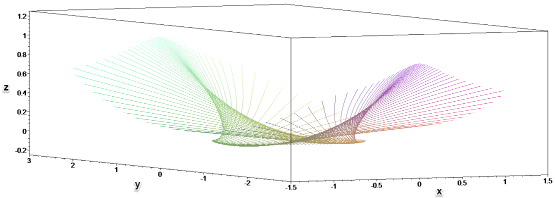
The GTG approach to gravitational waves IV

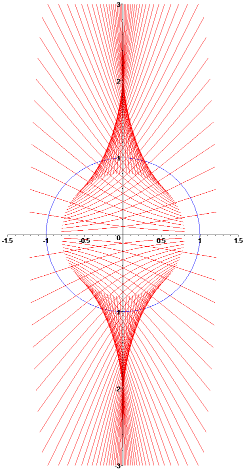
- So how does our version of gravitational waves compare with the conventional approach?
- Turns out it gives a quite different view of them (the 'gauge' is very different from what's used conventionally), and since it is exact, even recovers some physics that had been ignored or misunderstood in the conventional treatments
- This concerns something called 'velocity memory'
- Conventionally, after a wave passes through e.g. a ring of particles, they return to where they were



The GTG approach to gravitational waves V

- In our approach, find that the wave imparts a net velocity to the test particle that persists after the wave has passed
- Direction of motion depends on initial position in (x, y) plane versus polarization angle — get some rather beautiful patterns, and formation of caustics





- These are entirely absent in the standard approach — the linearisation loses them, so most astrophysicists didn't know this occurred!

- The effect has now been discovered via a more conventional approach, using exact solutions, by another group, working independently, with their first publication coinciding with when I first talked about the effect
- (See Zhang, P.-M., Duval, C., Gibbons, G. W., and Horvathy, P. A.: The memory effect for plane gravitational waves, Physics Letters B **772**, 743 (2017), arXiv: 1704.05997, and subsequent papers)
- Great deal of research likely to concentrate in this area in the near future

Progression towards electroweak and strong forces I

- Now want to progress towards the other two forces — electroweak and strong
- Starting point in this approach is Dirac spinor ψ
- We saw gravitational forces arise by gauging rotor transformations of ψ at the left

$$\psi \mapsto R\psi$$

where R is a Lorentz rotor, and from demanding invariance under remappings of spacetime (translations)

- Latter lead to the $\overline{h}(a)$ vector functions we discussed above, which make derivatives covariant (use $\overline{h}(\nabla)$ instead of ∇)
- In this approach, electroweak forces arise from gauging rotor transformations of ψ at the right

$$\psi \mapsto \psi R$$

and invariance under pseudoscalar transformations of the form

$$\psi \mapsto \psi e^{\alpha I}$$

Spinors I

- A useful item we need now are idempotents!
- Consider $\boxed{P_+ = \frac{1}{2}(1+\sigma_3)}$
- We have $(1+\sigma_3)^2 = 1+\sigma_3+\sigma_3+\sigma_3^2 = 2(1+\sigma_3)$, using $\sigma_3^2 = 1$, hence $P_+^2 = P_+$, i.e. is indeed an idempotent (or 'projector')
- Similarly, if $\boxed{P_- = \frac{1}{2}(1-\sigma_3)}$, then $P_-^2 = P_- = \frac{1}{2}(1-\sigma_3)$
- These idempotents enable us to create left and right handed Weyl spinors
- We can 'promote' a Pauli spinor ϕ to a Weyl one by multiplying by one of these: e.g. $\phi \mapsto \phi\frac{1}{2}(1+\sigma_3)$
- Note this means we can now apply a full Lorentz rotor at the left: $R\phi(1+\sigma_3)$
- Without the projector, this operation would take us outside the set of Pauli spinors, which only respond properly to spatial rotations, and wouldn't be covariant

- So we can now write a full Dirac spinor in terms of two Pauli spinors, ϕ and ω say, as

$$\psi = \phi\frac{1}{2}(1 - \sigma_3) - \omega I\sigma_2\frac{1}{2}(1 + \sigma_3)$$

  where the sign and $I\sigma_2$ in 2nd term turn out to be useful shortly. Any Dirac spinor can be written this way
- Note it has 8 (real) d.o.f., in accordance with it being the most general even element of the STA

Electroweak theory I

- We now have enough in hand to start on Electroweak theory!
- This works with two sectors: left and right
- The left sector is (for our purposes) Weyl spinors with the projector $P_{-} = \frac{1}{2}(1 - \sigma_3)$
- Right sector uses Weyl spinors of the $P_{+} = \frac{1}{2}(1 + \sigma_3)$ kind
- Neutrinos only appear in the left sector, and we call their wavefunction

$$\nu_L = -\nu_L^P \frac{1}{2}(1 - \sigma_3)$$

  where ν_L^P is a Pauli spinor
- For electrons, they have both a left and right sector, which we write

$$e_L = e_L^P \frac{1}{2}(1 - \sigma_3), \quad e_R = -e_R^P I\sigma_2 \frac{1}{2}(1 + \sigma_3)$$

Electroweak theory II

- Now, perhaps surprisingly, we combine the two left-hand components into a single Dirac wavefunction via

$$\psi_L = e_L \tfrac{1}{2}(1-\sigma_3) - \nu_L I\sigma_2 \tfrac{1}{2}(1+\sigma_3) = e_L^P \tfrac{1}{2}(1-\sigma_3) - \nu_L^P I\sigma_2 \tfrac{1}{2}(1+\sigma_3)$$

- The $I\sigma_2$ has converted the left-handed neutrino wavefunction into something which can be used as the right hand part of a full Dirac spinor, which combines the (left part of) the electron, and the neutrino
- From looking at the Lagrangian, we find that the symmetry we should look for is to find all multivectors N such that when $\psi_L \mapsto \psi_L e^N$, then $\psi_L \gamma_0 \tilde{\psi}_L$, the Dirac current, is invariant
- This picks out the set of bivectors which commute with γ_0, i.e. $I\sigma_1$, $I\sigma_2$ and $I\sigma_3$, and the pseudoscalar I, which reverses to itself, but anticommutes with γ_0
- The action of the $I\sigma_1$, $I\sigma_2$ and $I\sigma_3$ parts is thus a 'spatial rotor' R, which defines the $SU(2)$ part of the EW transformations, and the action of the pseudoscalar is like a 'phase rotation', so is $U(1)$

- Note: Nature seems only to take advantage of the $SU(2)$ symmetry for the left-handed wavefunctions
- The right-handed wavefunctions only see the $U(1)$ (duality transformations by the pseudoscalar) part
- Final object we need, is a way of coupling the left and right sectors of the theory together
- This is the role of the Higgs field
- Since our spinors are transforming on the right, we can form $SU(2)$-invariant inner products between them via products of the form

  $$\langle \theta \tilde{\psi} \rangle$$

  since under the rotor transformations this transforms to

  $$\langle \theta R(\tilde{R}\tilde{\psi}) \rangle = \langle \theta \tilde{\psi} \rangle$$

- The Higgs field is then a Pauli spinor H (i.e. a scaled, spatial rotor), which we use in the middle of such a product to provide a term which dynamically couples the left and right sectors:

$$\langle \psi_R H \tilde{\psi}_L \rangle$$

- Since the right-hand sector doesn't transform under the $SU(2)$ part, clear that H must provide the missing R and transform as

$$H \mapsto HR$$

  itself
- Looks as though this is going to be a problem (H forced to step outside being a Pauli spinor etc.), except the rotors here are purely spatial, so this is fine!
- Note that H is completely protected against actual Lorentz or spatial rotations in real space

- For one of these, we would have

$$\psi_R \mapsto R\psi_R, \quad \psi_L \mapsto R\psi_L$$

and so

$$\langle \psi_R H \tilde{\psi}_L \rangle \mapsto \langle R\psi_R H \tilde{\psi}_L \tilde{R} \rangle = \langle \psi_R H \tilde{\psi}_L \rangle$$

- So it's on this basis that we see H, the Higgs particle, is a Lorentz scalar
- It's actually a Pauli spinor, but doesn't respond to any spacetime transformations
- Don't have time for setting out the rest of the theory in STA, but in terms of forming a covariant derivative and field strength tensor things proceeds very much like what happens in 'Gauge Theory Gravity'
- Making the symmetries local corresponds to generating new forces, which are those of the elctroweak theory

- Some major differences are that transformations are on the right of the wavefunction, and only involve spatial transformations as regards the rotor R. Also have all the issues of setting up the correct couplings between left and right sectors
- Not all details done yet, but definitely clear can all be done in the STA
- How about strong forces?

Strong forces I

- These are currently being worked upon.
- A model initially due to my son (Robert Lasenby), extends the Dirac spinor ψ to be both a function of position x, and a linear function of a bivector argument B, so $\psi = \psi(B)$
- The separate colour fields then arise as $\psi_i = \psi(\sigma_i)$, where $\sigma_i = \gamma_i\gamma_0$, $i = 1, 2, 3$, are the unit norm spatial bivectors of the STA we discussed above
- $\psi = \psi(B, x)$, where x is position, is intriguingly like a spinor version of the Riemann tensor!
- Recently, a definite proposal for the form of these transformations of B has become clear, which gives a nice picture, within the STA, of what in the particle physics version is the $SU(3)$ group
- This is that we ask for transformations of bivectors F in the STA that keep the Hermitian inner product of F with itself invariant:

$$\langle \gamma_0 F \gamma_0 F \rangle$$

- (Remember Hermitian adjoint is reversion followed by 'reflection in time axis' and reversion for a bivector is just $F \mapsto -F$.)
- How does this work? We do this by considering double-sided operation on F
- Don't have time to go through the details, but form these from (for $i = 1, 2, 3$)

$$\begin{aligned} \hat{e}_i &= \text{ multiplication on the left by } \sigma_i\text{, so } \hat{e}_i(F) = \sigma_i F \\ \hat{f}_i &= \text{ multiplication on the right by } I\sigma_i\text{, so } \hat{f}_i(F) = FI\sigma_i \end{aligned} \tag{3}$$

and we claim the appropriate $U(n)$ generators are as follows.

$$\begin{aligned} \hat{E}_{ij} &= \hat{e}_i\hat{e}_j - \hat{f}_i\hat{f}_j, \quad i < j \\ \hat{F}_{ij} &= \hat{e}_i\hat{f}_j + \hat{e}_j\hat{f}_i, \quad i < j \\ \hat{J}_i &= \hat{e}_i\hat{f}_i, \quad i = 1, 2, 3 \end{aligned} \tag{4}$$

where there is no sum implied in the last line, i.e. each line contains three quantities, making up the expected 9 generators overall.

Strong forces III

- We restrict to $SU(3)$ as follows:
- Let us look at the sum of the $\hat{J}_i$,

$$\hat{J} = \hat{J}_1 + \hat{J}_2 + \hat{J}_3 \tag{5}$$

Since $\sigma_i F \sigma_i = -F$ (with a sum over the i), we have

$$\hat{J}(F) = -IF \tag{6}$$

for any bivector F. Thus $\hat{J}$ acts like the generator of a global 'phase rotation', but where the imaginary is the pseudoscalar.

- It is this part that is removed in making the transition to $SU(3)$ from $U(n)$
- Basically we have to take linear combinations of the $\hat{J}_i$ in which the overall sum is removed, i.e. the only combinations allowed are of the form

$$\alpha_1 \hat{J}_1 + \alpha_2 \hat{J}_2 + \alpha_3 \hat{J}_3, \quad \text{with} \quad \alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{7}$$

- This limits the number of independent generators to 8 instead of 9, giving the right number for $SU(3)$

Finite forms

A really nice feature of the approach is the ease with which we can derive finite forms of the transformations

- Looking at the two individual parts of the $\hat{E}_{ij}$ generators, i.e. $\hat{e}_i\hat{e}_j$, and $\hat{f}_i\hat{f}_j$, where $i < j$, it is clear that they mutually commute, and each squares to -1.
- Thus when we exponentiate $\hat{E}_{ij}$ to obtain a finite transformation, we can immediately write (with α a scalar)

$$\begin{aligned} \exp\left(\frac{\alpha}{2}\hat{E}_{ij}\right)(F) &= \exp\left(\frac{\alpha}{2}\hat{e}_i\hat{e}_j\right)\left(\exp\left(\frac{-\alpha}{2}\hat{f}_i\hat{f}_j\right)(F)\right) \\ &= \exp\left(\frac{\alpha}{2}\sigma_i\sigma_j\right)F\exp\left(\frac{\alpha}{2}\sigma_j\sigma_i\right) \\ &= R_{ij}F\tilde{R}_{ij} \end{aligned} \tag{8}$$

where R_{ij} is the spatial rotor $\exp\left(\frac{\alpha}{2}\sigma_i\sigma_j\right)$, which gives rotations through angle α about the $\epsilon_{ijk}\sigma_k$ axis.

- Thus if R is a general spatial rotor (and so has three d.o.f.), we can see that the $\hat{E}$ sector amounts to the set of spatial rotations $RF\tilde{R}$.
- For the $\hat{F}_{ij}$ generators, we again have that the two parts commute, and obtain

$$
\begin{aligned}
\exp\left(\frac{\alpha}{2}\hat{F}_{ij}\right)(F) &= \exp\left(\frac{\alpha}{2}\hat{e}_i\hat{f}_j\right)\left(\exp\left(\frac{\alpha}{2}\hat{e}_j\hat{f}_i\right)(F)\right) \\
&= \cos^2\frac{\alpha}{2}F + \tfrac{1}{2}I\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(\sigma_i F\sigma_j + \sigma_j F\sigma_i\right) \qquad (9) \\
&+ \sin^2\frac{\alpha}{2}\sigma_i\sigma_j F\sigma_j\sigma_i
\end{aligned}
$$

- To complete the set, the finite form for the $\hat{J}_i$ is

$$\exp\left(\alpha \hat{J}_i\right)(F) = \cos\alpha\, F + \sin\alpha\, I\sigma_i F \sigma_i \qquad (10)$$

with of course no sum on the r.h.s. For $\hat{J}$ we have

$$\exp\left(\alpha \hat{J}\right)(F) = e^{-\alpha I} F \qquad (11)$$

i.e. a global duality transformation

- Still a quite lot of work to do in this setup, but confident it can be done with wholly STA entities
- But: only dealing with one generation of particles, and there are 3!
- I'm fairly convinced that this will involve stepping outside the STA to things that are more like the CGA, and in particular the spaces either $Cl(4,1)$ (the '1d-up' CGA) or full CGA itself ($Cl(2,4)$ in application to spacetime)

Strong forces VII

- These may have the space we need for all the generations of particles, and possibly shed light on issues of why the left -hand and right-hand sectors are different in electroweak
- Anyway, good to know that e.g. an algebra that you are already using for the Conformal Geometric Algebra of 3d space ($Cl(4,1)$) may be the key to all the forces

Some credits and references I

Credits:

- GALD program which produced symmetric top animation by Christian Perwass
- Electromagnetic field-strength simulation for moving charge by Robert Lasenby

Further reading/references

Development of GA for Physics

- *Space-Time Algebra*, David Hestenes, Birkhauser (1966 originally, 2015 second edition)
- *Geometric Algebra for Physicists*, Chris Doran and Anthony Lasenby, CUP (2003)
- *Clifford Algebra to Geometric Calculus*, David Hestenes and Garret Sobczyk, D. Reidel (1984)
- *Geometric Algebra as a Unifying Language for Physics and Engineering and Its Use in the Study of Gravity*, Adv. Appl. Clifford Algebras, **27**, 733 (2017)

Some credits and references II

STA approach to Quantum Physics

- *Spacetime Algebra and Electron Physics*, Doran et al., Advances in Imaging and Electron Physics, **95**, 271 (1996)

Gravity and Gravitational Waves

- *Gravity, Gauge Theories and Geometric Algebra*, Anthony Lasenby, Chris Doran and Steve Gull, R. S. Lond. Philos. Trans. Ser. A, **356**, 487 (1998)
- *Geometric Algebra, Gravity and Gravitational Waves*, Anthony Lasenby, Adv. Appl. Clifford Algebras, **29**, 79 (2019)
- *The memory effect for plane gravitational waves*, Zhang, P.-M., Duval, C., Gibbons, G. W., and Horvathy, P. A., Physics Letters B **772**, 743 (2017)

Some credits and references III

Electroweak/Strong Forces

Published references are:

- David Hestenes, Space-time structure of weak and electromagnetic interactions, Foundations of Physics, Volume 12, Issue 2, pp.153-168 (1982)
- David Hestenes, Gauge Gravity and Electroweak Theory, in The Eleventh Marcel Grossmann Meeting On Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories, Berlin, 2008. pp. 629-647
- Also see arXiv:0807.0060 for this
- Doran & Lasenby, Geometric Algebra for Physicists, Chap 13, section 3,6 (CUP, 2003)

(Note these references don't explicitly cover the nature of the Higgs particle outlined here, or have an explicit model for the strong force. These covered in unpublished drafts by Anthony and Robert Lasenby) For a non-STA approach but still relevant to Clifford Algebra see papers by Cohl Furey, e.g. *Charge quantization from a number operator*, Cohl Furey, Phys.Lett.B, **742**, 195 (2015)