{ 2D Projective Geometric Algebra }

2D PGA Cheat Sheet SIGGRAPH 2019 Course Notes

Basics

Basis & Metric:

$$\mathbb{R}^*_{2,0,1}$$

	VECTOR			BIVECTOR			I= PSS
1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{01}$	$\mathbf{e}_{20}$	$\mathbf{e}_{12}$	$\mathbf{e}_{012}$
+1	0	+1	+1	0	0	-1	0
	LINE : ℓ			POINT : P			

Multiplication Table:

1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{01}$	$\mathbf{e}_{20}$	$\mathbf{e}_{12}$	$\mathbf{e}_{012}$
$\mathbf{e}_0$	0	$\mathbf{e}_{01}$	-$\mathbf{e}_{20}$	0	0	$\mathbf{e}_{012}$	0
$\mathbf{e}_1$	-$\mathbf{e}_{01}$	**1**	$\mathbf{e}_{12}$	-$\mathbf{e}_0$	$\mathbf{e}_{012}$	$\mathbf{e}_2$	$\mathbf{e}_{20}$
$\mathbf{e}_2$	$\mathbf{e}_{20}$	-$\mathbf{e}_{12}$	**1**	$\mathbf{e}_{012}$	$\mathbf{e}_0$	-$\mathbf{e}_1$	$\mathbf{e}_{01}$
$\mathbf{e}_{01}$	0	$\mathbf{e}_0$	$\mathbf{e}_{012}$	0	0	-$\mathbf{e}_{20}$	0
$\mathbf{e}_{20}$	0	$\mathbf{e}_{012}$	-$\mathbf{e}_0$	0	0	$\mathbf{e}_{01}$	0
$\mathbf{e}_{12}$	$\mathbf{e}_{012}$	-$\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{20}$	-$\mathbf{e}_{01}$	$-\mathbf{1}$	-$\mathbf{e}_0$
$\mathbf{e}_{012}$	0	$\mathbf{e}_{20}$	$\mathbf{e}_{01}$	0	0	-$\mathbf{e}_0$	0

Operators:

$\mathbf{ab}$		Geometric Product	
$\mathbf{a}^*$		Dual	
$\mathbf{a}^\perp$	$\mathbf{a}\mathrm{I}$	Polar	
$\tilde{\mathbf{a}}$		Reverse	
$\langle\mathbf{a}\rangle_n$		Select grade n	
$\mathbf{a}\wedge\mathbf{b}$	$\langle\mathbf{ab}\rangle_{\mathbf{s+t}}$	Outer Product	meet
$\mathbf{a}\vee\mathbf{b}$	$(\mathbf{a}^*\wedge\mathbf{b}^*)^*$	Regressive Product	join
$\mathbf{a}\cdot\mathbf{b}$	$\langle\mathbf{ab}\rangle_{\mathbf{\lvert s-t\rvert}}$	Inner Product	
$\mathbf{a}\times\mathbf{b}$	$\frac{1}{2}(\mathbf{ab}-\mathbf{ba})$	Commutator Product	
	$\mathbf{a}\mathbf{b}\tilde{\mathbf{a}}$	Sandwich Product	

Dual, Reverse:

Multivector	$a+b\mathbf{e_0}+c\mathbf{e_1}+d\mathbf{e_2}+e\mathbf{e_{01}}+f\mathbf{e_{20}}+g\mathbf{e_{12}}+h\mathbf{e_{012}}$
Dual	$h+g\mathbf{e^0}+f\mathbf{e^1}+e\mathbf{e^2}+d\mathbf{e^{01}}+c\mathbf{e^{20}}+b\mathbf{e^{12}}+a\mathbf{e^{012}}$
Reverse	$a+b\mathbf{e_0}+c\mathbf{e_1}+d\mathbf{e_2}-e\mathbf{e_{01}}-f\mathbf{e_{20}}-g\mathbf{e_{12}}-h\mathbf{e_{012}}$

Sub-algebras:

$\{1\}$	$\mathbb{R}$	Real	$\{1,\mathbf{e_{12}}\}$	$\mathbb{C}$	Complex
$\{1,\mathbf{e_0}\}$	$\mathbb{D}$	Dual	$\{1,\mathbf{e_1}\}$	$\mathbb{D}$	Hyperbolic
$\{1,\mathbf{e_{12}}\}$		rotors	$\{1,\mathbf{e_{01}},\mathbf{e_{20}}\}$		translators
$\{1,\mathbf{e_{01}},\mathbf{e_{20}},\mathbf{e_{12}}\}$		motors			

Geometry

Points, Lines:

Euclidean point at (x,y)	$x\mathbf{e}_{20}+y\mathbf{e}_{01}+\mathbf{e}_{12}$
Direction (ideal point) (x,y)	$x\mathbf{e}_{20}+y\mathbf{e}_{01}$
Line with eq. $a\mathbf{x}+b\mathbf{y}+c=0$	$\boldsymbol{\ell}=a\mathbf{e}_1+b\mathbf{e}_2+c\mathbf{e}_0$

Incidence:

Join points $\mathbf{P}_1,\mathbf{P}_2$ in line $\boldsymbol{\ell}$	$\boldsymbol{\ell}=\mathbf{P}_1\vee\mathbf{P}_2$
Meet lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$ in point P	$P=\boldsymbol{\ell}_1\wedge\boldsymbol{\ell}_2$

Project, Reject:

Line orthogonal to line $\boldsymbol{\ell}$, through point P	$\boldsymbol{\ell}\cdot\mathbf{P}=\boldsymbol{\ell}\times\mathbf{P}$
Project point $\mathbf{P}$ on line $\boldsymbol{\ell}$	$(\boldsymbol{\ell}\cdot\mathbf{P})\boldsymbol{\ell}$
Project line $\boldsymbol{\ell}$ on point $\mathbf{P}$	$(\boldsymbol{\ell}\cdot\mathbf{P})\mathbf{P}$
Direction orthogonal to line $\boldsymbol{\ell}$	$\boldsymbol{\ell}^\perp:=\boldsymbol{\ell}\mathbf{I}$

Norms and numerical values:

Euc. norm of $\boldsymbol{\ell}=c\mathbf{e}_0+a\mathbf{e}_1+b\mathbf{e}_2$:	$\lVert\boldsymbol{\ell}\rVert:=\sqrt{\boldsymbol{\ell}^2}(=\sqrt{a^2+b^2})$
Euc. norm of $\mathbf{P}=x\mathbf{e}_{20}+y\mathbf{e}_{01}+z\mathbf{e}_{12}$:	$\lVert\mathbf{P}\rVert:=\sqrt{\mathbf{P}\tilde{\mathbf{P}}}\ (=\sqrt{z^2})$
Ideal norm of ideal $\mathbf{P}=x\mathbf{e}_{20}+y\mathbf{e}_{01}$:	$\lVert\mathbf{P}\rVert_\infty:=\sqrt{x^2+y^2}$
Norm of motor $\mathbf{m}$	$\lVert\mathbf{m}\rVert:=\sqrt{\mathbf{m}\tilde{\mathbf{m}}}$
Numerical value of ideal $\boldsymbol{\ell}=c\mathbf{e}_0$:	$\lVert\boldsymbol{\ell}\rVert_\infty:=c$
Numerical value of pseudoscalar $a\mathbf{I}$	$\lVert a\mathbf{I}\rVert_\infty=a$

Metric:

Distance between points $\mathbf{P}_1,\mathbf{P}_2$	$\lVert\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2\rVert,\ \lVert\hat{\mathbf{P}}_1\times\hat{\mathbf{P}}_2\rVert_\infty$
Angle of intersecting lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$	$\cos^{-1}(\hat{\boldsymbol{\ell}}_1\cdot\hat{\boldsymbol{\ell}}_2),\ \sin^{-1}(\lVert\hat{\boldsymbol{\ell}}_1\wedge\hat{\boldsymbol{\ell}}_2\rVert)$
Distance parallel lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$	$\lVert\hat{\boldsymbol{\ell}}_1\wedge\hat{\boldsymbol{\ell}}_2\rVert_\infty$
Oriented dist. eucl. $\mathbf{P}$ to line $\boldsymbol{\ell}$	$\hat{\mathbf{P}}\vee\hat{\boldsymbol{\ell}},\ \lVert\hat{\mathbf{P}}\wedge\hat{\boldsymbol{\ell}}\rVert_\infty$
Angle betw. ideal $\mathbf{P}$ and line $\boldsymbol{\ell}$	$\sin^{-1}\lVert\hat{\mathbf{P}}\wedge\hat{\boldsymbol{\ell}}\rVert_\infty$
Angle bisector of $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$	$(\hat{\boldsymbol{\ell}}_1+\hat{\boldsymbol{\ell}}_2)$ or $\hat{\boldsymbol{\ell}}_1-\hat{\boldsymbol{\ell}}_2$
Perp. bisector of $\mathbf{P}_1$ and $\mathbf{P}_2$	$(\hat{\mathbf{P}}_1+\hat{\mathbf{P}}_2)(\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2)$
Altitudes of $\Delta\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$	$(\mathbf{P}_1\vee\mathbf{P}_2)\cdot\mathbf{P}_3$, etc.

Motors

Rotors & Translators:

Rotator α around point $\mathbf{P}_E$	$e^{\frac{\alpha}{2}\mathbf{P}_E}=\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\mathbf{P}_E$
Translator d orthogonal to $\mathbf{P}_\infty$	$e^{\frac{d}{2}\mathbf{P}_\infty}=1+\frac{d}{2}\mathbf{P}_\infty$
Motor between lines $\boldsymbol{\ell}_1$,$\boldsymbol{\ell}_2$	$\sqrt{\hat{\boldsymbol{\ell}}_2\hat{\boldsymbol{\ell}}_1}$
Logarithm of motor $\mathbf{m}$	$\widehat{\langle\mathbf{m}\rangle}_2$

Compose & Apply:

Compose motors $\mathbf{m}_1$ and $\mathbf{m}_2$	$\mathbf{m}_2\mathbf{m}_1$
Normalize motor $\mathbf{m}$	$\hat{\mathbf{m}}=\dfrac{\mathbf{m}}{\lVert\mathbf{m}\rVert}$
Square root of motor $\mathbf{m}$	$\sqrt{\mathbf{m}}=(1+\hat{\mathbf{m}})$
Reflect element $\mathbf{X}$ in line $\boldsymbol{\ell}$	$\boldsymbol{\ell}\mathbf{X}\boldsymbol{\ell}$
Transform $\mathbf{X}$ with motor $\mathbf{m}$	$\mathbf{m}\mathbf{X}\tilde{\mathbf{m}}$

More

Areas:

Area of $\Delta\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$	$\frac{1}{2}(\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2\vee\hat{\mathbf{P}}_3)$
Length of closed loop $\mathbf{P}_1\mathbf{P}_2...\mathbf{P}_n$	$\sum_{i=1}^{n-1}\lVert\hat{\mathbf{P}}_i\vee\hat{\mathbf{P}}_{i+1}\rVert$
Area of closed loop $\mathbf{P}_1\mathbf{P}_2...\mathbf{P}_n$	$\frac{1}{2}\lVert(\sum_{i=1}^{n-1}\hat{\mathbf{P}}_i\vee\hat{\mathbf{P}}_{i+1})\rVert_\infty$

Rigid Body Mechanics: (Valid in euclidean, elliptic & hyperbolic planes)

Kinematics-points, dynamics-lines	linear+angular unified
Element in the body/space frame	$\mathbf{x}_b/\mathbf{x}_s$
Path of $\mathbf{x}$ under the motion $\mathbf{g}$	$\mathbf{x}_s=\mathbf{g}\mathbf{x}_b\widetilde{\mathbf{g}},\ \mathbf{x}_b=\widetilde{\mathbf{g}}\mathbf{x}_s\mathbf{g}$
Velocity $\mathbf{V}_b$ in the body	$\mathbf{V}_b=\tilde{\mathbf{g}}\dot{\mathbf{g}}$ (a bivector)
Inertia tensor $\mathsf{A}:\bigwedge^2\leftrightarrow\bigwedge^1$	maps vel. $\leftrightarrow$ mom. in body
Momentum line $\mathbf{m}_b$ in the body	$\mathbf{m}_b=\mathsf{A}(\mathbf{V}_b)$
Kinetic energy E	$E=\mathbf{m}_b\vee\mathbf{V}_b$
Euler Eq. of Motion 1:	$\dot{\mathbf{g}}=\mathbf{g}\mathbf{V}_b$
Euler EoM 2: ($\mathbf{f}_b$ = ext. forces)	$\dot{\mathbf{V}}_b=2\mathsf{A}^{-1}(\mathbf{f}_b+(\mathbf{m}_b\times\mathbf{V}_b))$
Time derivative of energy E	$\dot{E}=-2\mathbf{f}_b\vee\mathbf{V}_b$
Work $w(t)=E(t)-E(0)$	$=\int_0^t\dot{E}ds=-2\int_0^t\mathbf{f}_b\vee\mathbf{V}_b ds$