{ 3D Projective Geometric Algebra }

3D PGA Cheat Sheet   SIGGRAPH 2019 Course Notes

Basics

Basis & Metric: [1]

$$\mathbb{R}^*_{3,0,1}$$

	VECTOR				BIVECTOR						TRIVECTOR				I = PSS
1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{01}$	$\mathbf{e}_{02}$	$\mathbf{e}_{03}$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{021}$	$\mathbf{e}_{013}$	$\mathbf{e}_{032}$	$\mathbf{e}_{123}$	$\mathbf{e}_{0123}$
+1	0	+1	+1	+1	0	0	0	−1	−1	−1	0	0	0	−1	0
	PLANE $\mathbf{p}$				LINE $\boldsymbol{\ell}$						POINT $\mathbf{P}$				

Multiplication Table:

1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{01}$	$\mathbf{e}_{02}$	$\mathbf{e}_{03}$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{021}$	$\mathbf{e}_{013}$	$\mathbf{e}_{032}$	$\mathbf{e}_{123}$	I
$\mathbf{e}_0$	0	$\mathbf{e}_{01}$	$\mathbf{e}_{02}$	$\mathbf{e}_{03}$	0	0	0	$-\mathbf{e}_{021}$	$-\mathbf{e}_{013}$	$-\mathbf{e}_{032}$	0	0	0	I	0
$\mathbf{e}_1$	$-\mathbf{e}_{01}$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_0$	$\mathbf{e}_{021}$	$-\mathbf{e}_{013}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{e}_{123}$	$\mathbf{e}_{02}$	$-\mathbf{e}_{03}$	I	$\mathbf{e}_{23}$	$\mathbf{e}_{032}$
$\mathbf{e}_2$	$-\mathbf{e}_{02}$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_{021}$	$-\mathbf{e}_0$	$\mathbf{e}_{032}$	$-\mathbf{e}_1$	$\mathbf{e}_{123}$	$\mathbf{e}_3$	$-\mathbf{e}_{01}$	I	$\mathbf{e}_{03}$	$\mathbf{e}_{31}$	$\mathbf{e}_{013}$
$\mathbf{e}_3$	$-\mathbf{e}_{03}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{e}_{013}$	$-\mathbf{e}_{032}$	$-\mathbf{e}_0$	$\mathbf{e}_{123}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	I	$\mathbf{e}_{01}$	$-\mathbf{e}_{02}$	$\mathbf{e}_{12}$	$\mathbf{e}_{021}$
$\mathbf{e}_{01}$	0	$\mathbf{e}_0$	$-\mathbf{e}_{021}$	$\mathbf{e}_{013}$	0	0	0	$\mathbf{e}_{02}$	$-\mathbf{e}_{03}$	I	0	0	0	$-\mathbf{e}_{032}$	0
$\mathbf{e}_{02}$	0	$\mathbf{e}_{021}$	$\mathbf{e}_0$	$-\mathbf{e}_{032}$	0	0	0	$-\mathbf{e}_{01}$	I	$\mathbf{e}_{03}$	0	0	0	$-\mathbf{e}_{013}$	0
$\mathbf{e}_{03}$	0	$-\mathbf{e}_{013}$	$\mathbf{e}_{032}$	$\mathbf{e}_0$	0	0	0	I	$\mathbf{e}_{01}$	$-\mathbf{e}_{02}$	0	0	0	$-\mathbf{e}_{021}$	0
$\mathbf{e}_{12}$	$-\mathbf{e}_{021}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	$-\mathbf{e}_{02}$	$\mathbf{e}_{01}$	I	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$\mathbf{e}_0$	$\mathbf{e}_{032}$	$-\mathbf{e}_{013}$	$-\mathbf{e}_3$	$-\mathbf{e}_{03}$
$\mathbf{e}_{31}$	$-\mathbf{e}_{013}$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$\mathbf{e}_{03}$	I	$-\mathbf{e}_{01}$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_{032}$	$\mathbf{e}_0$	$\mathbf{e}_{021}$	$-\mathbf{e}_2$	$-\mathbf{e}_{02}$
$\mathbf{e}_{23}$	$-\mathbf{e}_{032}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	I	$-\mathbf{e}_{03}$	$\mathbf{e}_{02}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$\mathbf{e}_{013}$	$-\mathbf{e}_{021}$	$\mathbf{e}_0$	$-\mathbf{e}_1$	$-\mathbf{e}_{01}$
$\mathbf{e}_{021}$	0	$\mathbf{e}_{02}$	$-\mathbf{e}_{01}$	$-I$	0	0	0	$\mathbf{e}_0$	$\mathbf{e}_{032}$	$-\mathbf{e}_{013}$	0	0	0	$\mathbf{e}_{03}$	0
$\mathbf{e}_{013}$	0	$-\mathbf{e}_{03}$	$-I$	$\mathbf{e}_{01}$	0	0	0	$-\mathbf{e}_{032}$	$\mathbf{e}_0$	$\mathbf{e}_{021}$	0	0	0	$\mathbf{e}_{02}$	0
$\mathbf{e}_{032}$	0	$-I$	$\mathbf{e}_{03}$	$-\mathbf{e}_{02}$	0	0	0	$\mathbf{e}_{013}$	$-\mathbf{e}_{021}$	$\mathbf{e}_0$	0	0	0	$\mathbf{e}_{01}$	0
$\mathbf{e}_{123}$	$-I$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{032}$	$\mathbf{e}_{013}$	$\mathbf{e}_{021}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_{03}$	$-\mathbf{e}_{02}$	$-\mathbf{e}_{01}$	-1	$\mathbf{e}_0$
I	0	$-\mathbf{e}_{032}$	$-\mathbf{e}_{013}$	$-\mathbf{e}_{021}$	0	0	0	$-\mathbf{e}_{03}$	$-\mathbf{e}_{02}$	$-\mathbf{e}_{01}$	0	0	0	$-\mathbf{e}_0$	0

Operators: [7]

$\mathbf{ab}$		Geometric Product	
$\mathbf{a}^*$		Dual [2]	
$\mathbf{a}^\perp$	$\mathbf{aI}$	Polarity	
$\tilde{\mathbf{a}}$		Reverse [3]	
$\hat{\mathbf{a}}$		Normalization [4]	
$\langle\mathbf{a}\rangle_\mathbf{n}$		Select grade n	
$\mathbf{a}\wedge\mathbf{b}$	$\langle\mathbf{ab}\rangle_{\mathbf{s+t}}$	Outer Product	meet
$\mathbf{a}\vee\mathbf{b}$	$(\mathbf{a}^*\wedge\mathbf{b}^*)^*$	Regressive Product	join
$\mathbf{a}\cdot\mathbf{b}$	$\langle\mathbf{ab}\rangle_{\mathbf{\lvert s-t\rvert}}$	Inner Product	
$\mathbf{a}\times\mathbf{b}$	$\frac{1}{2}(\mathbf{ab}-\mathbf{ba})$	Commutator Product	
	$\mathbf{ab\tilde{a}}$	Sandwich Product	

Dual, Reverse: [2] [3]

	1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{01}$	$\mathbf{e}_{02}$	$\mathbf{e}_{03}$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{021}$	$\mathbf{e}_{013}$	$\mathbf{e}_{032}$	$\mathbf{e}_{123}$	$\mathbf{I}$
MV	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
MV*	p	o	n	m	l	k	j	i	h	g	f	e	d	c	b	a
$\widetilde{\mathrm{MV}}$	a	b	c	d	e	$-f$	$-g$	$-h$	$-i$	$-j$	$-k$	$-l$	$-m$	$-n$	$-o$	p

Sub-algebras:

$\{1\}$ $\mathbb{R}$ Real

$\{1,\mathbf{e}_0\}$ $\mathbb{D}$ Dual

$\{1,\mathbf{e}_{12},\mathbf{e}_{31},\mathbf{e}_{23}\}$ $\mathbb{H}$ Quaternions / rotors

$\{1,\mathbf{e}_{01},\mathbf{e}_{02},\mathbf{e}_{03}\}$ translators

$\{1,\mathbf{e}_{12},\mathbf{e}_{31},\mathbf{e}_{23},\mathbf{e}_{01},\mathbf{e}_{02},\mathbf{e}_{03},\mathbf{I}\}$ Dual Quaternions / motors

$\{1,\mathbf{e}_{12}\}$ $\mathbb{C}$ Complex

$\{1,\mathbf{e}_1\}$ $\mathbb{D}$ Hyperbolic

Geometry

Points, Lines, Planes :

- Euclidean point (x,y,z): $\mathbf{P} = x\mathbf{e}_{032} + y\mathbf{e}_{013} + z\mathbf{e}_{021} + \mathbf{e}_{123}$
- Ideal point (direction) (x,y,z): $\mathbf{P} = x\mathbf{e}_{032} + y\mathbf{e}_{013} + z\mathbf{e}_{021}$
- Plane $ax+by+cz+d=0$: $\mathbf{p} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_0$

Incidence: [5]

- Join points/directions $\mathbf{P}_1,\mathbf{P}_2$ in line $\boldsymbol{\ell}$: $\boldsymbol{\ell} = \mathbf{P}_1\vee\mathbf{P}_2$
- Meet planes $\mathbf{p}_1,\mathbf{p}_2$ in line $\boldsymbol{\ell}$: $\boldsymbol{\ell} = \mathbf{p}_1\wedge\mathbf{p}_2$
- Join points $\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3$ in plane $\mathbf{p}$: $\mathbf{p} = \mathbf{P}_1\vee\mathbf{P}_2\vee\mathbf{P}_3$
- Meet planes $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ in point $\mathbf{P}$: $\mathbf{P} = \mathbf{p}_1\wedge\mathbf{p}_2\wedge\mathbf{p}_3$
- Join line $\boldsymbol{\ell}$ and point $\mathbf{P}$ in plane $\mathbf{p}$: $\mathbf{p} = \boldsymbol{\ell}\vee\mathbf{P}$
- Meet line $\boldsymbol{\ell}$ and plane $\mathbf{p}$ in point $\mathbf{P}$: $\mathbf{P} = \boldsymbol{\ell}\wedge\mathbf{p}$

Project, Reject:

- Plane $\perp$ to plane $\mathbf{p}$ through line $\boldsymbol{\ell}$: $\mathbf{p}\cdot\boldsymbol{\ell}$
- Line $\perp$ to plane $\mathbf{p}$ through point $\mathbf{P}$: $\mathbf{p}\cdot\mathbf{P}$
- Plane $\perp$ to line $\boldsymbol{\ell}$ through point $\mathbf{P}$: $\boldsymbol{\ell}\cdot\mathbf{P}$
- Project plane $\mathbf{p}$ onto point $\mathbf{P}$ [6]: $(\mathbf{p}\cdot\mathbf{P})\mathbf{P}$
- Project point $\mathbf{P}$ onto plane $\mathbf{p}$: $(\mathbf{p}\cdot\mathbf{P})\mathbf{p}$
- Project plane $\mathbf{p}$ onto line $\boldsymbol{\ell}$: $(\mathbf{p}\cdot\boldsymbol{\ell})\boldsymbol{\ell}$
- Project line $\boldsymbol{\ell}$ onto plane $\mathbf{p}$: $(\mathbf{p}\cdot\boldsymbol{\ell})\mathbf{p}$
- Project line $\boldsymbol{\ell}$ onto point $\mathbf{P}$: $(\boldsymbol{\ell}\cdot\mathbf{P})\mathbf{P}$
- Project point $\mathbf{P}$ onto line $\boldsymbol{\ell}$: $(\boldsymbol{\ell}\cdot\mathbf{P})\boldsymbol{\ell}$
- Direction $\perp$ to plane $\mathbf{p}$: $\mathbf{p}^\perp = \mathbf{pI}$
- Ideal line $\perp$ to line $\boldsymbol{\ell}$: $\boldsymbol{\ell}^\perp = \boldsymbol{\ell}\mathbf{I}$

Metric relations: (Some 2D-similar formulas omitted.)

- Distance of points $\mathbf{P}_1,\mathbf{P}_2$: $\|\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2\|$, $\|\hat{\mathbf{P}}_1\times\hat{\mathbf{P}}_2\|_\infty$
- Angle of inters. planes $\mathbf{p}_1,\mathbf{p}_2$: $\cos^{-1}(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)$, $\sin^{-1}(\|\hat{\mathbf{p}}_1\wedge\hat{\mathbf{p}}_2\|)$
- Dist. between $\parallel$ planes $\mathbf{p}_1,\mathbf{p}_2$: $\|\hat{\mathbf{p}}_1\wedge\hat{\mathbf{p}}_2\|_\infty$
- Angle of plane $\mathbf{p}$ and line $\boldsymbol{\ell}$: $\sin^{-1}(\|\langle\hat{\mathbf{p}}\hat{\boldsymbol{\ell}}\rangle_3\|)$
- Dist. between parallel $\mathbf{p}$ and $\boldsymbol{\ell}$: $(\|\langle\hat{\mathbf{p}}\hat{\boldsymbol{\ell}}\rangle_3\|_\infty)$
- Oriented distance $\mathbf{P}$ to $\mathbf{p}$: $\hat{\mathbf{P}}\vee\hat{\mathbf{p}}$, $\|\hat{\mathbf{P}}\wedge\hat{\mathbf{p}}\|_\infty$
- Oriented distance $\mathbf{P}$ to $\boldsymbol{\ell}$: $\|\hat{\mathbf{P}}\vee\hat{\boldsymbol{\ell}}\|$
- Angle bisector of $\mathbf{p}_1$ and $\mathbf{p}_2$: $(\hat{\mathbf{p}}_1+\hat{\mathbf{p}}_2)$ or $(\hat{\mathbf{p}}_1-\hat{\mathbf{p}}_2)$
- Common normal line to $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$: $\widehat{\boldsymbol{\ell}_1\times\boldsymbol{\ell}_2}$
- Angle α between $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$: $\alpha = \cos^{-1}(\hat{\boldsymbol{\ell}}_1\cdot\hat{\boldsymbol{\ell}}_2)$
- Distance between $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$: $d_{\boldsymbol{\ell}_1\boldsymbol{\ell}_2} = \csc\alpha\,(\hat{\boldsymbol{\ell}}_1\vee\hat{\boldsymbol{\ell}}_2)$

Norms & Motors

Norms: (Planes, points, and pss like 2D analogs) [8]

- Line $\boldsymbol{\ell} = \ldots + d\mathbf{e}_{12} + e\mathbf{e}_{31} + f\mathbf{e}_{23}$: $\|\boldsymbol{\ell}\| := \sqrt{\boldsymbol{\ell}\tilde{\boldsymbol{\ell}}} = \sqrt{d^2+e^2+f^2}$
- Ideal $\boldsymbol{\ell} = a\mathbf{e}_{01} + b\mathbf{e}_{02} + c\mathbf{e}_{03}$: $\|\boldsymbol{\ell}\|_\infty := \sqrt{a^2+b^2+c^2}$
- Normalize line $\boldsymbol{\ell}$: $\hat{\boldsymbol{\ell}} = \frac{\boldsymbol{\ell}}{\|\boldsymbol{\ell}\|}$ (eucl.) or $\frac{\boldsymbol{\ell}}{\|\boldsymbol{\ell}\|_\infty}$ (ideal)

Sandwiches and motors:

- Rotator α around line $\boldsymbol{\ell}_E$: $e^{\frac{\alpha}{2}\boldsymbol{\ell}_E} = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\boldsymbol{\ell}_E$
- Translator $d\perp$ to $\boldsymbol{\ell}_\infty$: $e^{\frac{d}{2}\boldsymbol{\ell}_\infty} = 1 + \frac{d}{2}\boldsymbol{\ell}_\infty$
- Screw axis $\boldsymbol{\ell}$ + pitch $p = \dfrac{d}{\alpha}$: $e^{t(1+p\mathbf{I})\boldsymbol{\ell}} = e^{t\boldsymbol{\ell}}e^{tp\mathbf{I}\boldsymbol{\ell}} = e^{tp\mathbf{I}\boldsymbol{\ell}}e^{t\boldsymbol{\ell}}$
- Motor between lines $\boldsymbol{\ell}_1,\boldsymbol{\ell}_2$: $\sqrt{\hat{\boldsymbol{\ell}}_2\hat{\boldsymbol{\ell}}_1} = \widehat{1+\hat{\boldsymbol{\ell}}_2\hat{\boldsymbol{\ell}}_1}$
- Logarithm of motor $\mathbf{m}$ [9]: $\mathbf{b} = \langle\mathbf{m}\rangle_2,\ s = \sqrt{-\mathbf{b}\cdot\mathbf{b}},\ p = \frac{-\mathbf{b}\wedge\mathbf{b}}{2s}$
  $\log\mathbf{m} = \left(\tan^{-1}\left(\frac{s}{\langle\mathbf{m}\rangle_0}\right) + \frac{p}{\langle\mathbf{m}\rangle_0}\right)\mathbf{b}\frac{s-p}{s^2}$

Compose & Apply:

- Compose motors $\mathbf{m}_1$ and $\mathbf{m}_2$: $\mathbf{m}_2\mathbf{m}_1$
- Normalize motor $\mathbf{m}$: $\hat{\mathbf{m}} = \frac{\mathbf{m}}{\|\mathbf{m}\|}$
- Square root simple motor $\mathbf{m}$: $\sqrt{\mathbf{m}} = \widehat{(1+\mathbf{m})}$
- Square root general motor $\mathbf{m}$: $\sqrt{\mathbf{m}} = (1+\mathbf{m})(1+\widehat{\langle\mathbf{m}\rangle} - \frac{1}{2}\langle\mathbf{m}\rangle_4)$
- Reflect element $\mathbf{X}$ in plane $\mathbf{p}$: $\mathbf{pXp}$
- Transform $\mathbf{X}$ with motor $\mathbf{m}$: $\mathbf{mX\tilde{m}}$

More

Volumes and areas:

- Volume of tetra. $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3\mathbf{P}_4$: $\frac{1}{6}(\hat{\mathbf{P}}_1\vee\hat{\mathbf{P}}_2\vee\hat{\mathbf{P}}_3\vee\hat{\mathbf{P}}_4)$
- Circum./area of edge loop [10]: $c = \sum\|\boldsymbol{\ell}_i\|,\quad a = \frac{1}{2!}\|\sum\boldsymbol{\ell}_i\|_\infty$
- Area/vol of triangle mesh [11]: $a = \frac{1}{2!}\sum\|\mathbf{f}_i\|,\quad v = \frac{1}{3!}\|\sum\mathbf{f}_i\|_\infty$

Rigid body mechanics: (Valid in euclidean, elliptic & hyperbolic space)

- Velocity, momentum (lin. + ang.!): bivectors $\boldsymbol{v}, \boldsymbol{m}$
- Element in the body/space frame: $\mathbf{x}_b/\mathbf{x}_s$
- Path of $\mathbf{x}$ under the motion $\mathbf{g}$: $\mathbf{x}_s = \mathbf{g}\mathbf{x}_b\tilde{\mathbf{g}},\ \mathbf{x}_b = \tilde{\mathbf{g}}\mathbf{x}_s\mathbf{g}$
- Velocity $\boldsymbol{v}_b$ in the body: $\boldsymbol{v}_b = \tilde{\mathbf{g}}\dot{\mathbf{g}}$
- Inertia tensor $\mathsf{A}: \bigwedge^2 \to \bigwedge^2$: $\boldsymbol{m}_b = \mathsf{A}(\boldsymbol{v}_b),\ \boldsymbol{v}_b = \mathsf{A}^{-1}(\boldsymbol{m}_b)$
- Kinetic energy E: $E = \boldsymbol{m}_b\vee\boldsymbol{v}_b$
- Euler Eq. of Motion 1: $\dot{\mathbf{g}} = \mathbf{g}\boldsymbol{v}_b$
- Euler EoM 2: ($\boldsymbol{f}_b$ = ext. forces): $\dot{\boldsymbol{v}}_b = 2\mathsf{A}^{-1}(\boldsymbol{f}_b + (\boldsymbol{m}_b\times\boldsymbol{v}_b))$
- Time derivative of energy E: $\dot{E} = -2\boldsymbol{f}_b\vee\boldsymbol{v}_b$
- Work $w(t) = E(t) - E(0)$: $= \int_0^t \dot{E}\,ds = -2\int_0^t \boldsymbol{f}_b\vee\boldsymbol{v}_b\,ds$

FOOTNOTES

1. Euclidean, Elliptic, Hyperbolic space: By choosing different values for $\mathbf{e}_0^2$ you obtain PGA also for elliptic and hyperbolic metric spaces. Many formulas on this sheet also apply to these spaces; the differences can be traced back to the differences in the ideal elements.

$\mathbb{R}^*_{3,0,1}$ - Euclidean PGA

	VECTOR				BIVECTOR						TRIVECTOR				I = PSS
1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{01}$	$\mathbf{e}_{02}$	$\mathbf{e}_{03}$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{021}$	$\mathbf{e}_{013}$	$\mathbf{e}_{032}$	$\mathbf{e}_{123}$	$\mathbf{e}_{0123}$
$+1$	0	$+1$	$+1$	$+1$	0	0	0	-1	-1	-1	0	0	0	-1	0
	PLANE $\mathbf{p}$				LINE$\boldsymbol{\ell}$						POINT $\mathbf{P}$				

$\mathbb{R}^*_{4,0,0}$ - Elliptic PGA

	VECTOR				BIVECTOR						TRIVECTOR				I = PSS
1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{01}$	$\mathbf{e}_{02}$	$\mathbf{e}_{03}$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{021}$	$\mathbf{e}_{013}$	$\mathbf{e}_{032}$	$\mathbf{e}_{123}$	$\mathbf{e}_{0123}$
$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	$+1$
	PLANE $\mathbf{p}$				LINE $\boldsymbol{\ell}$						POINT $\mathbf{P}$				

$\mathbb{R}^*_{3,1,0}$ - Hyperbolic PGA

	VECTOR				BIVECTOR						TRIVECTOR				I = PSS
1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{01}$	$\mathbf{e}_{02}$	$\mathbf{e}_{03}$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{021}$	$\mathbf{e}_{013}$	$\mathbf{e}_{032}$	$\mathbf{e}_{123}$	$\mathbf{e}_{0123}$
$+1$	-1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1	$+1$	$+1$	$+1$	-1
	PLANE $\mathbf{p}$				LINE $\boldsymbol{\ell}$						POINT $\mathbf{P}$				

2. Duality: See § 5.10 of the Course Notes.

3. Reverse: The reverse $\widetilde{\mathbf{X}}$ of an element $\mathbf{X}$ is the element obtained by reversing all the products of 1-vectors that occur in it.

4. Normalize: The normalization operator $\widetilde{\mathrm{x}}$ does different things, depending on its argument; they all have in common that the result is *normalized* in the category it belongs to. Typically a normalized element $\mathbf{n}$ satisfies $\mathbf{n}\widetilde{\mathbf{n}} = \pm\mathbf{1}$.

5. Intersecting lines: See § 8.1.2 of the Course Notes.

6. Remarks on projection: See § 7.2 of the Course Notes.

7. Outer and Inner product: s and t are the grades of $\mathbf{a}$ and $\mathbf{b}$, respectively.

8. Ideal norm: See § 7.1 of the Course Notes.

9. Logarithm of a motor: if $s = 0$, the motor is a pure translation and its logarithm $\log \mathbf{m} = \frac{\mathbf{m}}{\langle \mathbf{m} \rangle_0} - 1$. Else if $\langle \mathbf{m} \rangle_0 = 0$, the motor is a *turn* with logarithm $\log \mathbf{m} = \frac{\pi}{2} - \frac{\langle \mathbf{m} \rangle_4}{s}$. See § 8.1.6 of the Course Notes.

10. Edge loop: the edges (lines) $\boldsymbol{\ell}_i$ of an edge loop are found by joining adjacent normalized points, $\boldsymbol{\ell}_i = \widehat{\mathbf{P}}_i \vee \widehat{\mathbf{P}}_{i+1}$, where the $n+1$-th point is the same as the first (the area formula works for edge loops contained in a single plane).

11. Triangle mesh: the faces (planes) $\mathbf{f}_i$ of a triangle mesh are found by joining the three points of each triangle (with consistent winding order), $\mathbf{f}_i = \widehat{\mathbf{P}}_{i1} \vee \widehat{\mathbf{P}}_{i2} \vee \widehat{\mathbf{P}}_{i3}$.

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IMPLEMENTATION

C++, C#, Rust, Python and javascript implementations, updated course notes and cheat-sheets on **bivector.net**